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Abstract: In this work, we consider the time-harmonic Galbrun's equation under spherical symmetry in the context of the wave propagation in the Sun without flow and rotation, and neglecting the perturbations to the gravitational potential. The model parameters are taken from the solar model `S` for the interior of the Sun, and we introduce the model `AtmoCAI` (ideal atmospheric behavior with constant adiabatic index) to extend them into the atmosphere. This atmospheric extension is based on the model `Atmo` used for the scalar wave propagation where, in addition, we assume a constant adiabatic index in the atmosphere. Due to the spherical symmetry, by writing the original equation in a vector spherical harmonic basis, we obtain the ODE for the modal radial and tangential coefficients of the unknown displacements. We then construct the outgoing modal solutions, the 3D Green's kernel, and radiation boundary conditions. The construction is justified by indicial and asymptotic analysis of the modal radial ODE. While the singular set in the presence of attenuation only consists of the origin, our analysis shows that without attenuation, there are also other singular points which, however, have positive indicial exponents. Our asymptotic analysis makes appear the correct wavenumber and the high-order terms of the oscillatory phase function, which we use to characterize outgoing solutions. The radiation boundary conditions are built for the modal radial ODE and then derived for the initial equation. We approximate them under different hypothesis and propose some formulations that are independent of the horizontal wavenumber and can thus easily be applied for 3D problems.

Key-words: vectorial helioseismology, Galbrun's equation, outgoing solutions, radiation boundary conditions, indicial analysis, Green tensor, Whittaker functions, Coulomb potential

Solutions sortantes et conditions de radiation pour l'équation des ondes vectorielles en héliosismologie avec un modéle d'atmosphère idéal

Résumé : Dans ce travail, nous considérons l'équation harmonique de Galbrun en symétrie sphérique pour la propagation d'ondes dans le soleil, sans flot ni rotation, et en négligeant les perturbations du potentiel de gravité. Les paramètres sont extraits du modèle `S` pour l'intérieur du soleil, et nous introduisons un modèle `AtmoCAI` (comportement atmosphérique idéal avec un indice adiabatique constant) pour leur extension dans l'atmosphère solaire. Cette extension est basée sur le modèle `Atmo` utilisé dans le cas scalaire, que nous enrichissons en prenant l'indice adiabatique constant. De part la symétrie sphérique, en écrivant le problème dans une base harmonique sphérique vectorielle, nous obtenons l'EDO modale pour les coefficients radiaux et tangentiels du déplacement. Nous construisons les solutions sortantes modales, le noyau de Green en 3D et obtenons des conditions aux limites de radiation. La construction est motivée par l'analyse indicielle et asymptotique de l'EDO radiale modale. En présence d'atténuation, la seule singularité est á l'origine, alors que dans le cas sans atténuation, nous identifions les autres singularités qui, cependant, ont un exposant indiciel positif. Notre analyse asymptotique fait apparaître un nombre d'onde approprié et les termes d'ordres élevés de la phase, qui nous servent à caractériser les solutions sortantes. Les conditions aux limites de radiation sont construites pour l'EDO modale puis étendues au problème initial. Nous les approximons sous différentes hypothèses et proposons plusieurs approximations independantes du nombre d'onde horizontal qui peuvent être facilement utilisées pour les problèmes 3D.

Mots-clés : équations vectorielles pour l'héliosismologie, équantion de Galbrun, solutions sortantes, conditions aux limites absorbantes, analyse indicielle, tenseur de Green, fonctions de Whittaker, potentiel de Coulomb

Contents

1 Introduction

In this work, we consider a version of the time-harmonic Galbrun's equation under spherical symmetry for applications in helioseismology. We carry out indicial and asymptotic analysis in order to define outgoing solutions and radiation boundary conditions. The adiabatic wave motion due to a time-harmonic perturbation of frequency $\omega/(2\pi)$ is modeled by the vector field $\boldsymbol{\xi}(\mathbf{x},\omega) \in \mathbb{R}^3$ on top of a stationary background (i.e., the physical parameters do not vary with the time), and solves the following form of the Galbrun's equation,

$$-\rho_0\left(\omega^2 + 2\mathrm{i}\,\omega\,\Gamma\right)\boldsymbol{\xi} + \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) + \rho_0(\boldsymbol{\xi}\cdot\nabla)\nabla\Phi_0 = \mathbf{f} \quad \text{in } \mathbb{R}^3\,, \tag{1.1}$$

where we omit the space dependency for clarity. Here, Φ_0 denotes the background gravity potential that satisfies,

$$\Delta\Phi_0 = 4\,\pi\,G\,\rho_0\,, \tag{1.2}$$

where G is the gravitational constant. The operator $\boldsymbol{\mathcal{P}}$ is defined as

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} = -\nabla\left[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\right] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) - \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] + (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0\,, \tag{1.3}$$

Here, we ignore flows, rotation and the perturbations due to the gravitational potential using the Cowling approximation [14]. The stationary background is characterized as a hydrodynamical system and is given in terms of the following scalar quantities that depend only on the position $\mathbf{x}$: the density ρ_0, the pressure p_0, the attenuation Γ and the adiabatic index γ, and the background is further assumed to be spherically symmetric. While we have in mind solar applications, (1.1) is also employed in aeroacoustics to describe the propagation of the acoustic sound produced by an aircraft engine in the presence of (air) flow around this engine, see [33, 24]. One particularity of (1.1) is that the unknown $\boldsymbol{\xi}$ is a Lagrangian displacement in terms of an Eulerian variable $\mathbf{x}$, such that this formulation is also called Eulerian-Lagrangian description of the perturbation. The Galbrun's equation was introduced by Galbrun in [19] in the 1930s, and re-derived by Poirée in [30]; we refer to [33] for a brief history of the equation and to [24] for a more recent derivation in the context of aeroacoustics. The derivation of the equation in the context of stellar physics is given by [26, Eqn 28-30].

For application in helioseismology, the unknown $\boldsymbol{\xi}(\mathbf{x},\omega)$ represents the displacement of the solar material observed at the (geometrical) position $\mathbf{x} \in \mathbb{R}^3$ in the photosphere (the Sun's surface layer), cf. [21, Section 2.3], [8, Eqn 4–5] and [22, Eqn 5]. The projection along a line-of-sight $\hat{\mathbf{l}} \in \mathbb{R}^3$ of the solution $\boldsymbol{\xi}$ (corresponding to a stochastic source $\mathbf{f}$), $-\mathrm{i}\,\omega\,\boldsymbol{\xi}\cdot\hat{\mathbf{l}}$, simulates the line-of-sight Dopplergrams which are crucial input data for time-distance helioseismology, cf. [22, Eqn 15] and [9, Eqn 3]. In our specific problem, up to 500 km above the "surface" of the Sun, the background quantities are given by the model `S` of [12]. To establish to model parameters in the atmosphere after this height of 500 km, we introduce the `AtmoCAI` model (ideal atmospheric behavior with constant adiabatic index) which is built from the model `Atmo` employed in [17, 3, 5, 6] for the atmosphere when studying a scalar wave equation for helioseismology. While the model `Atmo` determines the extension of the parameters such that (i) the sound speed c_0 and attenuation Γ are constant in the atmosphere, (ii) the density ρ_0 is exponentially decaying in the atmosphere, the vectorial equation needs additional specifications to extend p_0, γ and Φ_0', see our discussion in Section 3.

While the oscillations in the Sun are driven by stochastic convection below the surface, the main input for time-distance helioseismology is the expectation value of the cross-covariance which is a deterministic quantity that can be estimated numerically from the deterministic Green's kernel of (1.1). The main task is to numerically compute a suitable Green's function of (1.1). To complicate the matter, the phenomenon occurs in an infinite spatial domain. Depending on the purposes of the analysis, suitable boundary conditions have to be imposed to obtain uniqueness of solutions, which also comes as a necessity for numerical resolution based on domain discretization, which requires the finiteness of domain. With this general goal discussed, there are three main groups of results obtained in our work, which also serve as its novelty.

1. We provide a global analysis of the resulting radial ODE. In particular, we carry out a detailed indicial analysis of the coefficients both in the presence and absence of attenuation, and an asymptotic analysis at infinity. We show that the singular sets are all *regular singular*. While the singular set in the presence of attenuation only consists of the origin, our analysis shows that without attenuation, there are also

other singular points which however have positive indicial exponents, cf. Table 1. Only the indicial roots at the origin were studied in the litterature by [35] (for $\ell > 0$) under more restrictive hypothesis on the background coefficients. The asymptotic analysis provides a means to define outgoing solution, and characterizes the oscillator behavior of such a solution. In particular, our analysis makes appear an appropriate wavenumber (given in (9.12)) and the high-order terms of the oscillatory phase function (see (9.22) and (9.27)).

2. We construct the modal radial Green's kernel, from which we obtain the outgoing 3D Green's kernel in (9.89). This quantity is the main ingredient to compute Born sensitivity kernels that are used in helioseismology in order to interpret the observations. See for example [21, 8] for a discussion in Cartesian geometry (small patch of the Sun) and [9] in spherical.

3. Regarding boundary conditions, our work extends the RBCs from [5, 6, 4] to the vectorial case in a spherically symmetric background, and thus also benefits from the ODE techniques of [2]. We obtain low-order radiation boundary conditions both in modal form (i.e. for the coefficient of the decomposition in vector spherical harmonics) and in 3D form. They are built by approximating the square root of the operator and keeping the terms up to order r^{-2}. However, we noticed numerically that including the gravity term that is of order r^{-3} greatly improves the accuracy of the boundary conditions. Physically, surface gravity waves (f-mode) are located in the first megameters below the surface while acoustic modes (p-mode) propagate deeper in the solar interior. It could explain the importance of this term in the quality of the approximate boundary conditions. We propose several boundary conditions under the hypothesis of small wavenumber or small angle of incidence by including or not the gravity term.

Before the work of [20, 17], a common practice was to impose a free-surface boundary condition at the surface of the Sun (Lagrangian pressure perturbation vanishes on the surface) [10], which is adequate for describing trapped waves and waves at low-frequencies, but is however not suitable for high-frequencies at which waves can propagate to infinity. To describe the infinite phenomenon, most results in helioseismology work with a scalar wave equation with a new unknown $\mathfrak{u} = \rho_0 c_0^2 \nabla \cdot \boldsymbol{\xi}$, that solves,

$$-\nabla \cdot \left(\frac{1}{\rho_0} \nabla \mathfrak{u} \right) \; - \; \frac{\sigma^2}{\rho_0 \, c_0^2} \mathfrak{u} \; = \; \text{source term} \,, \tag{1.4}$$

where c_0 denotes the sound speed and $\sigma^2 = (\omega^2 + 2\mathrm{i}\,\omega\,\Gamma)$. This is obtained from (1.1) under simplifying assumptions, cf. [20, 17, 3, 5, 6]. [20] uses the model `Atmo` to extend the model `S` up to 4 Mm above the surface of the Sun, allowing the use of the Sommerfeld condition, $\partial_n \mathfrak{u} = \mathrm{i}\,\sigma/c_\infty\,\mathfrak{u}$, thus with wavenumber σ/c_∞ at the boundary, with c_∞ the constant value of the sound speed in the atmosphere. This approach increases the computational domain and thus the computational cost. Also working with the scalar equation (1.4), new radiation boundary conditions (RBC) are constructed in [3] under the `Atmo` extension, which allow for an exponential decay of the background density in the atmosphere, and placement of the artificial boundary right after the end of the model `S` (500 km above the surface). These conditions are called non-local, small-angle approximation (SAI) and high-frequency (HF) approximation. The non-local and SAI results directly from the factorization of the scalar wave operator and provide satisfactory results. On the other hand, the HF families are obtained from approximation of the non-local condition with $(\sigma/c_0)^{-1}$ as the small quantity. Low-order HF conditions, while offering lower precision, have the advantage of being readily implementable in 3D or non-spherical geometry and even in the time-domain, without the intervention of tangential derivatives. We also refer to [23] for another technique to truncate the discretization domain using perfectly matched layers, which is also the approach taken by [24, 15] for the vectorial equation.

While the theoretical question of characterizing outgoing solutions is interesting in its own right, understanding the asymptotic behavior of the exact outgoing solution allows to calculate the correct dominant oscillatory behavior of the solution. In particular, solutions are approximately described at infinity by spherical waves whose propagation speed is described by a pertinent wavenumber. Working with the right wavenumber has direct repercussion in high-order approximations of the non-local condition, i.e., the HF families. The radiation conditions in [17, 3] are justified theoretically in [5, 6, 1] by using long-range scattering theory for Schrödinger equation. Additionally, [5, 6] also identify the appropriate wavenumbers which in fact also depends on the density scale height in the atmosphere, and thus offers a better performance for high-frequency approximations of the non-local condition, as shown in [5, 6, 4].

In a different setting, working with the vectorial equations, based on the work of [18, section 3.3 p. 89] in the absence of flow, [34, 33, 29] apply a non-reflective condition, which takes the form of an impedance condition relating the Lagrangian pressure perturbation to the normal direction of the displacement, and is equivalent to $\nabla \cdot \boldsymbol{\xi} = \mathrm{i}(\sigma/\mathrm{c}_\infty)\, \boldsymbol{\xi} \cdot \mathbf{n}$, see Subsection 10.3.3 for further discussion. As mentioned above, this is the wavenumber associated to a uniform (i.e., constant) background, and this condition should work well under this assumption. However, in order to work with the `Atmo` assumptions, one would expect the wavenumber to also depend on the rate of decay in the atmosphere of the density background, as shown in our work.

The report is organized as follows. After some discussion on the Galbrun's equation and more precise statement of our working assumptions in Sections 2 and 3, we write the unknown displacement $\boldsymbol{\xi}$ in the vectorial spherical harmonic basis, and we obtain the decoupled systems in Section 4, cf. Proposition 3. These systems are solved by the radial and the tangential coefficients of $\boldsymbol{\xi}$ in the aforementioned basis. It is shown that the solution of the vectorial equation is completely determined by its radial coefficient. We also reduce the radial modal ODE into a Schrödinger form, called *conjugate modal radial equation*, which facilitates the construction of outgoing solution. In order to construct the outgoing solution, we carry out an analysis of singularity in Section 7 and an asymptotic analysis in Section 8 for the coefficients of the modal ODE. These ingredients are used in the construction of the modal Green's kernel in Subsection 9.1.4 and of the 3D Green's kernel in Subsection 9.2. The analysis of this section also makes appear the appropriate wavenumber, given by (9.12), which controls the oscillations of the outgoing solution at infinity (under `AtmoCAI` assumption). In Section 10, we follow the same procedure as in [5] to construct the nonlocal RBC and its low-order SAI and HF approximations. These are first obtained for the modal conjugate radial unknown in Subsection 10.1 and then obtained for the radial coefficients and tangential ones in Subsection 10.2. We also put these into 3D forms in Subsection 10.3, that can be applied in a direct 3D discretization of equation (1.1).

2 Notations

In this section, we review the main notation and quantities that are used throughout the report.

We denote by $\sqrt{}$ the square root branch that uses the argument branch $[0, 2\pi)$ while $(\cdot)^{1/2}$ uses the argument branch $(-\pi, \pi]$.

2.1 Physical parameters and scaled variables

We consider the propagation of time-harmonic waves in the Sun defined by

- $R_\odot$ is the Sun's radius, with approximation $R_\odot = 695.510 \times 10^6$ m.
- R is the (non-scaled) position along the Sun's radius.
- r is the scaled radius position such that $r = R/R_\odot$.
- r_a denotes the scaled position at which the solar atmosphere begins, given in the model `S` by $r_a = 1.000\,712\,6$, which corresponds to about 4.96×10^5 m above $R_\odot$.
- ω is the angular frequency.
- $\boldsymbol{\xi}$ is the Lagrange vectorial displacement perturbation.
- Φ_0 is the background gravity potential given from (1.2).

Therefore, in the following of the document, we mostly work with the scaled radius, such that $r = 1$ corresponds to the solar position $R_\odot$.

Next, the physical parameters are considered as *radial* quantities, only varying with the radius, they are extracted from the model `S` of [12].

- c_0 is the scaled background velocity (also referred to as sound speed or wave speed), shown in Figure 1a. Here, we work with the scaled quantity, $\mathrm{c}_0 = c_0/R_\odot$ given in s^{-1}, with c_0 the original solar sound speed in $\mathrm{m\,s}^{-1}$ given in model `S`.

- ρ_0 is the background density given in $\mathrm{kg\,m^{-3}}$.
- γ is the adiabatic index, in the model `S`, γ is not constant, except in the atmosphere and

$$1 < \gamma < 2\,, \tag{2.1}$$

as illustrated in Figure 3.

- Γ is the attenuation coefficient.
- p_0 is the scalar pressure field.
- G is the gravitational constant, $G = 6.674\,08 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$.
- σ is the *complex frequency* that encodes the attenuation such that

$$\sigma \,=\, \omega\sqrt{1 \,+\, 2\mathrm{i}\frac{\Gamma}{\omega}}\,. \tag{2.2}$$

We also refer Remark 13 for the dimensionless of the coefficients of our working ODE.

Attenuation model For the representation of Γ, it commonly follows the power law

$$\Gamma(\omega) \,=\, \Gamma_0\,\left|\frac{\omega}{\omega_0}\right|^2\beta\,, \tag{2.3}$$

with, cf. [20, Eqn. 79],

$$\frac{\Gamma_0}{2\pi} = 4.29\ \mathrm{\mu Hz}\,, \qquad \frac{\omega_0}{2\pi} = 3\ \mathrm{mHz}\,, \qquad \beta = 5.77\,. \tag{2.4}$$

As an alternative, one can also consider a constant attenuation model with

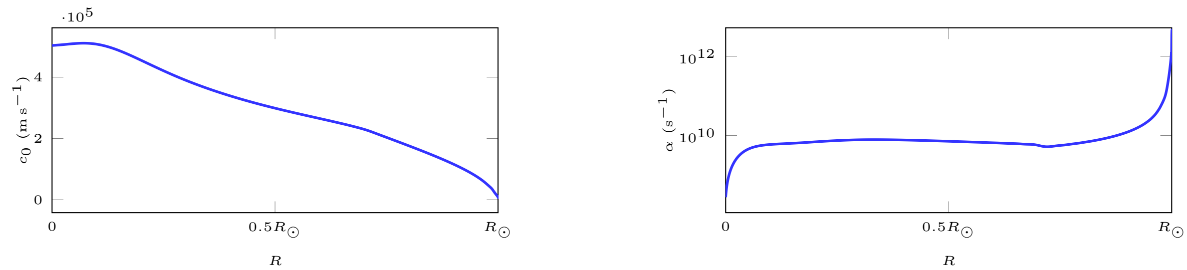
$$\frac{\Gamma}{2\pi} = 20\ \mathrm{\mu Hz}\,, \qquad \text{constant attenuation as an alternative to the power law.} \tag{2.5}$$

Scale height functions Scale height functions are defined as the derivative of the logarithmic of the model parameter. These are defined for the density, the velocity and the adiabatic index, such that

$$\begin{aligned} \alpha\,(=\alpha_{\rho_0}) \,&=\, -\frac{\rho_0'}{\rho_0}\,,\\ \alpha_{\mathrm{c}_0} \,&=\, -\frac{\mathrm{c}_0'}{\mathrm{c}_0}\,,\\ \alpha_\gamma \,&=\, -\frac{\gamma'}{\gamma}\,. \end{aligned} \tag{2.6}$$

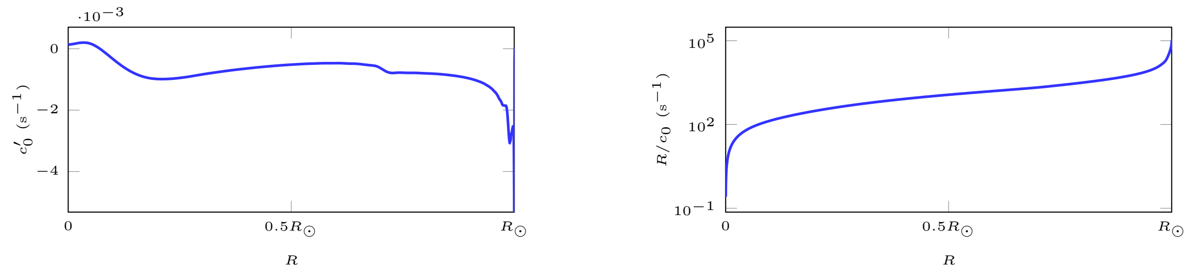
2.2 Numerical representation

The values of the physical parameters are given, point-wise, in the model `S` from [12]. Because we also need the derivative of the parameters (via the scale height), we first create a spline representation of the parameters, which are then explicitly defined, including their derivatives. Namely, from the couples (r, values) given in model `S`, we deduce a basis of cubic splines, where we guarantee that the error between the spline representation and the original model is less than 0.1%. The partition of the interval is not homogeneous, in order to have the slowest number of splines as possible. In Appendix F, we provide the resulting representation. In Figures 1 to 3, we illustrate the physical quantities. In particular, we observe that the profiles are relatively stable at the beginning, while drastic changes appear near the surface region.



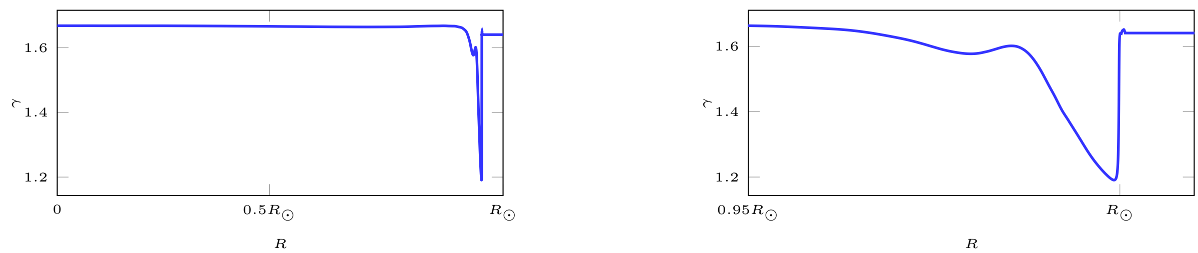
(a) Velocity profile from model S. (b) $\alpha = -\rho'/\rho$ profile from model S.

Figure 1: Profiles of the solar parameters with the model S. In our analysis, we consider the scaled velocity $\mathrm{c}_0 = c_0/R_\odot$, given in terms of the scaled radius $r = R/R_\odot$.



(a) Derivative of the velocity given by model S. (b) Profile of r/c_0 from model S.

Figure 2: Quantities used for the analysis associated with the solar model S. In our analysis, we consider the scaled velocity $\mathrm{c}_0 = c_0/R_\odot$, given in terms of the scaled radius $r = R/R_\odot$, therefore, we have $R/c_0 = r/\mathrm{c}_0$.



(a) Solar adiabatic index γ from model S. (b) Zoom near $R = R_\odot$ for the Solar adiabatic index γ

Figure 3: Adiabatic index profile extracted from the solar model S.

2.3 Surface operators and vector spherical harmonics

2.3.1 Coordinate systems and derivatives

The Cartesian basis is denoted by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The spherical basis is denoted $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, with $0 \leq \phi < 2\pi$ and $0 \leq \theta < \pi$, such that

$$\begin{aligned}
\mathbf{e}_r &= \sin\theta\cos\phi\;\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3\,,\\
\mathbf{e}_\theta &= \cos\theta\;\cos\phi\;\mathbf{e}_1 + \cos\theta\;\sin\phi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3\,,\\
\mathbf{e}_\phi &= -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2\,.
\end{aligned} \tag{2.7}$$

For a scalar f, we have

$$\begin{aligned}
\nabla f &= \partial_r f\,\mathbf{e}_r + \frac{\partial_\theta f}{r}\,\mathbf{e}_\theta + \frac{\partial_\phi f}{r\,\sin\theta}\,\mathbf{e}_\phi\\
\Delta f &= \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\,\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\,\sin\theta^2}\partial^2_\phi\,f\,.
\end{aligned} \tag{2.8}$$

For a vector $\mathbf{v}$, we have

$$\mathbf{v} \;=\; v_r\,\mathbf{e}_r \,+\, v_\theta\,\mathbf{e}_\theta \,+\, v_\phi\,\mathbf{e}_\phi\,, \tag{2.9}$$

$$\nabla\cdot\mathbf{v} \;=\; \frac{1}{r^2}\partial_r\big(r^2\,v_r\big) \,+\, \frac{\partial_\theta(\sin\theta\,v_\theta)}{r\sin\theta} \,+\, \frac{\partial_\phi v_\phi}{r\,\sin\theta}\,. \tag{2.10}$$

The material derivative is defined such that

$$\begin{aligned}(\mathbf{v}\cdot\nabla)\,\mathbf{F} \;=\; &\Big(\mathbf{v}\cdot\nabla F_r \,-\, \frac{v_\theta\,F_\theta}{r} \,-\, \frac{v_\phi\,F_\phi}{r}\Big)\mathbf{e}_r\\ &+\,\Big(\mathbf{v}\cdot\nabla F_\theta \,-\, \frac{v_\phi\,F_\phi}{r}\,\cot\theta \,+\, \frac{v_\theta\,F_r}{r}\Big)\mathbf{e}_\theta\\ &+\,\Big(\mathbf{v}\cdot\nabla F_\phi \,+\, \frac{v_\phi\,F_r}{r} \,+\, \frac{v_\phi\,F_\theta}{r}\,\cot\theta\Big)\mathbf{e}_\phi\end{aligned} \tag{2.11}$$

Remark 1. *Another way to think about material derivative is using*

$$\underbrace{(\mathbf{v}\cdot\nabla)\,\mathbf{F}}_{\textit{material derivative of }\mathbf{F}\textit{ along }\mathbf{v}} \;=\; \nabla\mathbf{F} \underbrace{\;\cdot\;}_{\textit{tensor contraction}} \mathbf{v}\,. \tag{2.12}$$

The gradient of a vector $\mathbf{F}$ *in spherical coordinates is*

$$\begin{aligned}\nabla\mathbf{F} \;=\; &\Big(\partial_r F_r\Big)\mathbf{e}_r\otimes\mathbf{e}_r \,+\, \Big(\frac{\partial_\theta F_r}{r} - \frac{F_\theta}{r}\Big)\mathbf{e}_r\otimes\mathbf{e}_\theta \,+\, \Big(\frac{\partial_\phi F_r}{r\sin\theta} \,-\, \frac{F_\phi}{r}\Big)\mathbf{e}_r\otimes\mathbf{e}_\phi\\ +\, &\Big(\partial_r F_\theta\Big)\mathbf{e}_\theta\otimes\mathbf{e}_r \,+\, \Big(\frac{\partial_\theta F_\theta}{r} + \frac{F_r}{r}\Big)\mathbf{e}_\theta\otimes\mathbf{e}_\theta \,+\, \Big(\frac{\partial_\phi F_\theta}{r\sin\theta} \,-\, \frac{F_\phi}{r}\cot\theta\Big)\mathbf{e}_\theta\otimes\mathbf{e}_\phi\\ +\, &\Big(\partial_r F_\phi\Big)\mathbf{e}_\phi\otimes\mathbf{e}_r \,+\, \Big(\frac{\partial_\theta F_\phi}{r}\Big)\mathbf{e}_\phi\otimes\mathbf{e}_\theta \,+\, \Big(\frac{\partial_\phi F_\phi}{r\sin\theta} \,+\, \frac{F_r}{r} \,+\, \frac{F_\theta}{r}\cot\theta\Big)\mathbf{e}_\phi\otimes\mathbf{e}_\phi\,.\end{aligned} \tag{2.13}$$

Contracting with $\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + v_\phi\mathbf{e}_\phi$ *to the right gives*

$$\begin{aligned}\nabla\mathbf{F}\cdot\mathbf{v} \;=\; &\Big(v_r\,\partial_r F_r \,+\, v_\theta\frac{\partial_\theta F_r}{r} - v_\theta\frac{F_\theta}{r} \,+\, v_\phi\frac{\partial_\phi F_r}{r\sin\theta} \,-\, v_\phi\frac{F_\phi}{r}\Big)\mathbf{e}_r\\ +\, &\Big(v_r\,\partial_r F_\theta \,+\, v_\theta\frac{\partial_\theta F_\theta}{r} \,+\, v_\theta\frac{F_r}{r} \,+\, v_\phi\frac{\partial_\phi F_\theta}{r\sin\theta} \,-\, v_\phi\frac{F_\phi}{r}\cot\theta\Big)\mathbf{e}_\theta\\ +\, &\Big(v_r\partial_r F_\phi \,+\, v_\theta\frac{\partial_\theta F_\phi}{r} \,+\, v_\phi\frac{\partial_\phi F_\phi}{r\sin\theta} \,+\, v_\phi\frac{F_r}{r} \,+\, v_\phi\frac{F_\theta}{r}\cot\theta\Big)\mathbf{e}_\phi\\ =\, &\Big((\mathbf{v}\cdot\nabla)F_r \,-\, v_\theta\frac{F_\theta}{r} \,-\, v_\phi\frac{F_\phi}{r}\Big)\mathbf{e}_r\\ +\, &\Big((\mathbf{v}\cdot\nabla)F_\theta \,+\, v_\theta\frac{F_r}{r} \,-\, v_\phi\frac{F_\phi}{r}\cot\theta\Big)\mathbf{e}_\theta\\ +\, &\Big((\mathbf{v}\cdot\nabla)F_\phi \,+\, v_\phi\frac{F_r}{r} \,+\, v_\phi\frac{F_\theta}{r}\cot\theta\Big)\mathbf{e}_\phi\,.\end{aligned} \tag{2.14}$$

△

2.3.2 Surface differential operators

The unit sphere is referred to by $\mathbb{S}^2$, along which we have

$$\nabla_{\mathbb{S}^2} f \;:=\; \partial_\theta f\,\mathbf{e}_\theta \,+\, \frac{\partial_\phi f}{\sin\theta}\,\mathbf{e}_\phi\,, \tag{2.15a}$$

$$\nabla_{\mathbb{S}^2}\cdot\mathbf{v} \;:=\; \frac{\partial_\theta(\sin\theta\,v_\theta)}{\sin\theta} \,+\, \frac{\partial_\phi v_\phi}{\sin\theta}\,, \tag{2.15b}$$

$$\Delta_{\mathbb{S}^2} f \;=\; \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta f) \,+\, \frac{1}{\sin\theta^2}\partial_\phi^2\,f\,, \tag{2.15c}$$

$$\mathbf{curl}_{\mathbb{S}^2} f \;:=\; -\mathbf{n}\times\nabla_{\mathbb{S}^2} f\,, \tag{2.15d}$$

$$\mathrm{curl}_{\mathbb{S}^2}\mathbf{v} \;=\; \mathbf{n}\cdot\mathbf{curl}\,\mathbf{v}\,. \tag{2.15e}$$

In addition, we have that

$$\begin{aligned}&\nabla_{\mathbb{S}^2}\cdot\mathbf{curl}_{\mathbb{S}^2} f \;=\; 0\,,\\ &\mathrm{curl}_{\mathbb{S}^2}\,\nabla_{\mathbb{S}^2} \;=\; 0\,,\\ &\Delta_{\mathbb{S}^2} \;=\; \nabla_{\mathbb{S}^2}\cdot\nabla_{\mathbb{S}^2} \;=\; -\mathrm{curl}_{\mathbb{S}^2}\,\mathbf{curl}_{\mathbb{S}^2}\,f\,.\end{aligned} \tag{2.16}$$

2.3.3 Spherical harmonics

The spherical basis is of dimension $2\ell+1$, cf. [28, 9.3.1] or [28, Theorem 9.11 p. 238], and comprises of

$$\mathrm{Y}_\ell^m(\theta,\phi) \,:=\, \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,\mathrm{P}_\ell^{|m|}(\cos\theta)\,e^{\mathrm{i}m\phi} \quad , \quad \ell\in\mathbb{N}, m=-\ell,\dots,\ell\,. \tag{2.17}$$

It has property

$$\Delta_{\mathbb{S}^2}\mathrm{Y}_\ell^m \;=\; -\ell(\ell+1)\,\mathrm{Y}_\ell^m. \tag{2.18}$$

Remark 2. *As in [28, Remark 9.12], we also use the notation*

$$\mathrm{Y}_\ell^m(\widehat{\mathbf{x}}) \tag{2.19}$$

for $\widehat{\mathbf{x}}$ the unit vector with spherical coordinates θ, ϕ. △

Vector spherical harmonics These are defined by, see [28, Eqn. 9.56, Section 9.3.3] or [27, Definition 3.336, p. 107],

$$\begin{aligned}
\mathbf{P}_\ell^m(\widehat{\mathbf{x}}) \;&=\; \mathrm{Y}_\ell^m(\widehat{x})\,\mathbf{e}_r\,,\\
\mathbf{C}_\ell^m(\widehat{\mathbf{x}}) \;&=\; \frac{1}{\sqrt{\ell(\ell+1)}}\,\mathbf{curl}_{\mathbb{S}^2}\mathrm{Y}_\ell^m \;=\; -\frac{1}{\sqrt{\ell(\ell+1)}}\,\mathbf{e}_r\times\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \quad , \quad \ell=1,2,\dots\,,\\
\mathbf{B}_\ell^m(\widehat{\mathbf{x}}) \;&=\; \frac{1}{\sqrt{\ell(\ell+1)}}\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \quad , \quad \ell=1,2,\dots\,.
\end{aligned} \tag{2.20}$$

We note that $\mathbf{C}_\ell^m$ can also be written as,

$$\mathbf{C}_\ell^m = -\mathbf{e}_r\times\mathbf{B}_\ell^m\,. \tag{2.21}$$

They have the following properties.

1. Properties with the divergence and the curl,

$$\nabla_{\mathbb{S}^2}\cdot\mathbf{C}_\ell^m = 0\,, \tag{2.22}$$

   and

$$\nabla_{\mathbb{S}^2}\cdot\mathbf{B}_\ell^m = \frac{1}{\sqrt{\ell(\ell+1)}}\nabla_{\mathbb{S}^2}\cdot\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m = \frac{1}{\sqrt{\ell(\ell+1)}}\Delta_{\mathbb{S}^2}\mathrm{Y}_\ell^m = -\sqrt{\ell(\ell+1)}\mathrm{Y}_\ell^m\,. \tag{2.23}$$

2. From their definitions, they are point-wise perpendicular, cf. [27, Eqn. 3.132].

3. The following set

$$\mathbf{C}_\ell^m\,,\,\mathbf{B}_\ell^m \quad , \quad \ell=1,2,\dots\,, \quad m=-\ell,\dots,\ell\,, \tag{2.24}$$

   forms a complete orthonormal basis for the set of tangent vectors[1] $L^2_t(\mathbb{S}^2)$, cf. [28, Lemma 9.15 p. 241].

4. Together with the fact that Y_ℓ^m for $\ell=0,1,\dots$ and $m=-\ell,\dots,\ell$ forms a complete orthonormal basis for $L^2(\mathbb{S}^2)$, cf. [28, 9.11],

$$\mathbf{P}_0^0\,,\,\mathbf{C}_\ell^m\,,\,\mathbf{B}_\ell^m\,,\,\ell=1,2,\dots\,, \quad m=-\ell,\dots,\ell \tag{2.26}$$

   forms a complete orthonormal basis for $L^2(\mathbb{S}^2)^3$, the space of surface vectors on the unit sphere $\mathbb{S}^2$ whose components are $L^2(\mathbb{S}^2)$.

[1] For a bounded domain with $\mathcal{C}^2$ connected boundary $\partial\Omega$, the space of surface tangential vector fields in $L^2(\partial\Omega)$ is defined as, cf. [28, Eqn. 3.13],

$$L^2_t(\partial\Omega) \,:=\, \Big\{\mathbf{u}\in L^2(\partial\Omega)^3\,|\,\nu_{\partial\Omega}\cdot\mathbf{u} \,=\, 0 \text{ almost everywhere on } \partial\Omega\Big\}\,. \tag{2.25}$$

3 Equations of motion for model S+AtmoCAI

We have introduced the main equation for the propagation of time-harmonic waves, in terms of the Lagrangian perturbation $\boldsymbol{\xi}$:

$$-\rho_0\left(\omega^2 + 2\mathrm{i}\,\omega\,\Gamma\right)\boldsymbol{\xi} + \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) + \rho_0(\boldsymbol{\xi}\cdot\nabla)\nabla\Phi_0 = \mathbf{f}\,, \qquad \text{in } \mathbb{R}^3, \tag{3.1}$$

with

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} = -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) - \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] + (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0\,, \tag{3.2}$$

and

$$\Delta\Phi_0 = 4\,\pi\,G\,\rho_0\,. \tag{3.3}$$

As mentioned in the introduction, we work in a context of helioseismplogy and take a simplification of the full Galbun's equation see Remark 3. We detail in Assumption 1 the simplification involved.

Remark 3 (Full Galbrun's equation)**.** *The original equation in time-harmonic regime from Lyndell-Bell's paper [26, Eqn. 17, 28-30], also called Galbrun's equation is*

$$\boxed{\rho_0\big(\mathrm{i}\sigma + \mathbf{v}_0\cdot\nabla\big)^2\boldsymbol{\xi} + \boldsymbol{\mathcal{R}}\boldsymbol{\xi} + \boldsymbol{\mathcal{P}}\boldsymbol{\xi} + \boldsymbol{\mathcal{G}}\boldsymbol{\xi} = \mathbf{f}\,,} \tag{3.4}$$

where $\boldsymbol{\xi}$ is the Lagrangian perturbation to the background, and

1. *the operator $\boldsymbol{\mathcal{P}}$ is defined as,*

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} = -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) - \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] + (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0\,, \tag{3.5}$$

2. *the complete gravity operator $\boldsymbol{\mathcal{G}}$ (in the opposite sign convention[2] to [26, Eqn. 23]) is defined as*

$$\boldsymbol{\mathcal{G}}\boldsymbol{\xi} = \rho_0\,\nabla\mathsf{S}(\boldsymbol{\xi}) + \rho_0\,(\boldsymbol{\xi}\cdot\nabla)\nabla(\Phi_0)\,, \tag{3.6}$$

with the perturbation in gravitational potential S satisfying the relation,

$$\Delta\,\mathsf{S}(\boldsymbol{\xi}) = -4\pi G\,\nabla\cdot(\rho_0\boldsymbol{\xi})\,, \tag{3.7}$$

and

$$\Delta\Phi_0 = 4\,\pi\,G\,\rho_0\,. \tag{3.8}$$

Since the fundamental solution for the Laplacian is $-\frac{1}{4\pi|x|}$, we have

$$\Phi_0(\mathbf{x}) = -G\int\frac{\rho_0(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}\,. \tag{3.9}$$

3. *The rotation operator $\boldsymbol{\mathcal{R}}$ around axis in Ω direction, is given as,*

$$\boldsymbol{\mathcal{R}}\boldsymbol{\xi} := \underbrace{2\rho_0\,\Omega\times(\mathrm{i}\omega + \mathbf{v}_0\cdot\nabla)\boldsymbol{\xi}}_{\textit{Coriolis force}} + \underbrace{\rho_0\,\Omega\times(\Omega\times\boldsymbol{\xi})}_{\textit{Centrifugal force}}\,. \tag{3.10}$$

△

Remark 4. *The following quantity denotes the Eulerian perturbations of the density, fluid pressure, and gravitational potential denoted respectively by $\delta^{\mathrm{E}}_{\rho}$, $\delta^{\mathrm{E}}_{\mathrm{p}}$, see, e.g., [24, Eqn 1.66,1.68 p.30], [11, Eqn. 3.44,3.45,3.41, 3.50 p. 50-51],*

$$\delta^{\mathsf{E}}_{\rho} = -\nabla\cdot(\rho_0\,\boldsymbol{\xi}) = -(\nabla\rho_0)\cdot\boldsymbol{\xi} - \rho_0\nabla\cdot\boldsymbol{\xi}\,; \tag{3.11a}$$

$$\delta^{\mathsf{E}}_{\mathrm{p}} = -\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 + \frac{\gamma\,\mathrm{p}_0}{\rho_0}\big(\delta^{\mathsf{E}}_{\rho} + \boldsymbol{\xi}\cdot\nabla\rho_0\big) = -\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 - \rho_0\,\mathrm{c}_0^2\,\nabla\cdot\boldsymbol{\xi}\,; \tag{3.11b}$$

$$\delta^{\mathsf{E}}_{\Phi} = \mathsf{S}(\boldsymbol{\xi}) \qquad\Rightarrow\qquad \Delta\,\delta^{\mathsf{E}}_{\Phi} = 4\,\pi\,G\,\delta^{\mathsf{E}}_{\rho}\,. \tag{3.11c}$$

△

[2] In [26, Eqn. 23], $\boldsymbol{\mathcal{G}}\boldsymbol{\xi} = -\rho_0\nabla\tilde{\mathsf{S}}(\boldsymbol{\xi}) - \rho_0\,(\boldsymbol{\xi}\cdot\nabla)\nabla\tilde{\phi}_0$ where $\tilde{\phi}_0 = -\Phi_0$ and $\tilde{\mathsf{S}} = -\mathsf{S}$.

3.1 Assumptions

In order to work with the problem made of (3.1) and (3.2) instead of the full equation (3.4), we have the following set of assumptions.

Assumption 1 (General assumption)**.** *In our study, we assume that*

1. *We assume that there is no rotation involved,* $\Omega = 0$*.*
2. *We work in a region where there is no flow, that is* $\mathbf{v}_0 = 0$*.*
3. *The first term in the gravity is ignored (i.e., no perturbation in gravity), such that we leave out* $\mathsf{S}(\boldsymbol{\xi})$*.*

From these three assumptions, (3.4) and (3.5) reduce to (3.1) and (3.2).

Assumption 2. *We further assume that,*

1. *the background parameters* ρ_0 *and* p_0 *have a radial dependence, thus*

$$\nabla \mathrm{p}_0 \;=\; (\partial_r \mathrm{p}_0)\, \mathbf{e}_r \;=\; \mathrm{p}_0'\, \mathbf{e}_r\,. \tag{3.12}$$

2. *The adiabatic equation of state for the parameters* ρ_0*,* p_0 *and* c_0 *with* γ *the adiabatic index is given by*

$$\mathrm{c}_0^2\, \rho_0 \;=\; \gamma\, \mathrm{p}_0\,. \tag{3.13}$$

3. *We assume that the external source* $\mathbf{f}$ *is of compact support.*

3.1.1 Representation in the interior of the Sun: model `S`

In the interior of the Sun, we follow model `S`, illustrated in Figures 1 and 2. The physical quantities ρ_0, c_0, p_0 and Φ_0 are radial, and the hydrostatic support reduces to

$$\mathrm{p}_0' \;=\; -\rho_0\, \Phi_0' \quad , \quad r \leq r_a\,. \tag{3.14}$$

See Subsection 3.3 and Remark 5 below for more discussion on the hydrostatic equilibrium in the interior. The background sound speed in model `S` sastifies the following assumption,

Assumption 3.

$$r \;\mapsto\; \frac{r}{\mathrm{c}_0(r)} \quad \textit{is strictly increasing on } [0\,,\, r_a]. \tag{3.15}$$

In fact, in the model `S`, the sound speed c_0 increases slightly close to the center of the Sun before it takes on a steep decrease as one moves towards the surface, see Figure 1. However, the function $r \mapsto r/\mathrm{c}_0(r)$ is strictly increasing on $[0\,,\, r_a]$ as observed in Figure 2.

3.1.2 Representation in the solar atmosphere: model `AtmoCAI`

We define the `AtmoCAI`, also called *ideal atmospheric behavior with constant adiabatic background.* At 500 km above the surface of the Sun, which coincides the end of the model `S`, the density ρ_0 is imposed to be exponentially decaying and the background sound speed c_0 and adiabatic index γ are extended by constants. The extension of the fluid pressure p_0 follows by maintaining the adiabatic condition in the atmosphere.

Definition 1 (Model `AtmoCAI`)**.** *In the atmosphere region, described by* $r \geq r_a$*, the model* `AtmoCAI` *defines the physical parameters such that:*

1. *the sound speed* c_0 *is constant,*
2. *the adiabatic coefficient* γ *is constant,*
3. ρ_0 *is exponentially decreasing, such that the density scale height* $\alpha_{\rho_0} (= \alpha)$ *is constant.*

Remark 5. *We note that the hydrostatic support, in particular in its radial version (3.14), is not compatible with the assumption that ρ_0 is exponentially decreasing. In particular in this case, equation (4.12) does not hold.* △

Remark 6 (model `AtmoHE`)**.** *We also have the option of assuming hydrostatic equilibrium in the atmosphere, at the expense of letting go the assumption of constant adiabatic index in the atmosphere. This model is called* ***AtmoHE****.* △

3.2 Equivalent forms of the operator $\boldsymbol{\mathcal{P}}$

Proposition 1. *The operator $\boldsymbol{\mathcal{P}}$ defined in (3.2) can also be written as*

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} = \nabla\big[(1-\gamma)\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] - \mathrm{p}_0\nabla(\nabla\cdot\boldsymbol{\xi}) - \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] + (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0 \tag{3.16a}$$

$$= -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) - \nabla^t\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 \tag{3.16b}$$

$$= -\nabla\cdot\boldsymbol{\tau}\,, \tag{3.16c}$$

with

$$\boldsymbol{\tau} := (\gamma-1)\,\mathrm{p}_0\,(\nabla\cdot\boldsymbol{\xi})\,\mathbb{I}_{3\times3} + \mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\,, \tag{3.17}$$

where $\mathbb{I}_{3\times3}$ is the identity matrix.

Here, (3.16a) corresponds to [26, Eq. (25)], and (3.16b) is [24, Eq (1.65), p.30].

Proof. Starting from (3.2), since $\mathrm{p}_0\nabla\nabla\cdot\boldsymbol{\xi} = \nabla(\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}) - (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi})$, we can rewrite

$$\nabla\big[(1-\gamma)\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] - \mathrm{p}_0\nabla(\nabla\cdot\boldsymbol{\xi}) = -\nabla\big[\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi})\,. \tag{3.18}$$

Next, we compare with [24, Eq. (1.65) p.30],

$$\begin{aligned}\nabla^t\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 = \sum_{i=1}^{3}\big(\partial_{x_j}\pi_i\boldsymbol{\xi}\big)\big(\partial_{x_i}\mathrm{p}_0\big) &= \sum_{i=1}^{3}\partial_{x_j}\big[(\pi_i\boldsymbol{\xi})(\partial_{x_i}\mathrm{p}_0)\big] - (\pi_i\boldsymbol{\xi})\big(\partial_{x_j}\partial_{x_i}\mathrm{p}_0\big)\\ &= \partial_{x_j}\sum_{i=1}^{3}(\pi_i\boldsymbol{\xi})(\partial_{x_i}\mathrm{p}_0) + \sum_{i=1}^{3}(\pi_i\boldsymbol{\xi})\big(\partial_{x_i}\partial_{x_j}\mathrm{p}_0\big)\,.\end{aligned}$$

In the second term, we have interchanged the order of differentiation, and use the definition of the material derivative in the Cartesian coordinates,

$$\begin{aligned}(\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0 &= \sum_{i=1}^{3}(\pi_i\boldsymbol{\xi})\big(\partial_{x_i}\partial_{x_j}\mathrm{p}_0\big) = \sum_{i=1}^{3}(\pi_i\boldsymbol{\xi})\big(\partial_{x_j}\partial_{x_i}\mathrm{p}_0\big)\\ \Rightarrow(\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0 &= \boldsymbol{\xi}\cdot\underbrace{\nabla\nabla}_{\text{Hessian}}\mathrm{p}_0 == (\nabla\nabla\mathrm{p}_0)\cdot\boldsymbol{\xi}\,.\end{aligned} \tag{3.19}$$

As a result, we have,

$$\nabla^t\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 = \nabla\big[\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0\big] - (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0\,, \tag{3.20}$$

and $\boldsymbol{\mathcal{P}}$ can be written as

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} = -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) - \nabla^t\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0\,. \tag{3.21}$$

We now define

$$\boldsymbol{\tau} := (\gamma-1)\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\,\mathbb{I}_{3\times3} + \mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\,, \tag{3.22}$$

where $\mathbb{I}_{3\times 3}$ is the 3×3 identity matrix. We consider $\nabla\cdot\boldsymbol{\tau}$, we have $(\nabla^t\boldsymbol{\xi})_{ij} = \partial_i\boldsymbol{\xi}_j$ and

$$\begin{aligned}
\nabla\cdot\big(\mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\big) &= \sum_{j=1}^{3}\partial_j\big(\mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\big)_{ij} = \sum_{j=1}^{3}\partial_j\big(\mathrm{p}_0\,\partial_i\boldsymbol{\xi}_j\big)\\
&= \sum_{j=1}^{3}\big(\partial_j\mathrm{p}_0\big)\,\partial_i\boldsymbol{\xi}_j + \sum_{j=1}^{3}\mathrm{p}_0\,\partial_j\big(\partial_i\boldsymbol{\xi}_j\big)\\
&= \big(\nabla^t\boldsymbol{\xi}\big)\cdot\nabla\mathrm{p}_0 + \mathrm{p}_0\nabla\cdot\nabla^t\boldsymbol{\xi}\\
&= \big(\nabla^t\boldsymbol{\xi}\big)\cdot\nabla\mathrm{p}_0 + \mathrm{p}_0\nabla\nabla\cdot\boldsymbol{\xi}\,.
\end{aligned}$$

In the last equality, we have assumed that $\boldsymbol{\xi}$ is regular enough for interchanging the order of integration, in order to obtain

$$\nabla\cdot\nabla^t\boldsymbol{\xi} = \sum_{j=1}^{3}\partial_j\big(\partial_i\boldsymbol{\xi}_j\big) = \sum_{j=1}^{3}\partial_i\big(\partial_j\boldsymbol{\xi}_j\big) = \nabla\nabla\cdot\boldsymbol{\xi}\,. \tag{3.23}$$

The above calculation gives

$$\nabla\cdot\boldsymbol{\tau} = \nabla\big((\gamma-1)\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big) + \nabla\cdot\big(\mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\big) = \nabla\big((\gamma-1)\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big) + (\nabla^t\boldsymbol{\xi})\cdot\nabla\mathrm{p}_0 + \mathrm{p}_0\nabla\nabla\cdot\boldsymbol{\xi}\,. \tag{3.24}$$

We next use (3.18), which gives

$$\nabla\big[(\gamma-1)\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] + \mathrm{p}_0\nabla(\nabla\cdot\boldsymbol{\xi}) = \nabla\big[\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] - (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi})\,, \tag{3.25}$$

to group together the first and third term in the right-hand-side of (3.24),

$$\nabla\cdot\boldsymbol{\tau} = \nabla\big[\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] - (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) + (\nabla^t\boldsymbol{\xi})\cdot\nabla\mathrm{p}_0\,.$$

As a result of the above calculation, we can write $\boldsymbol{\mathcal{P}}$ as

$$\boldsymbol{\mathcal{P}} = -\nabla\cdot\boldsymbol{\tau}\,,\qquad \text{with}\quad \boldsymbol{\tau} := (\gamma-1)\,\mathrm{p}_0\,(\nabla\cdot\boldsymbol{\xi})\,\mathbb{I}_{3\times 3} + \mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\,. \tag{3.26}$$

□

3.3 Hydrostatic equilibrium (for the interior)

Following[3] [11, Eqn. 3.30 p. 48], the equation of motion of a background at equilibrium without flow and with only gravity as external force reduces to the *equation of hydrostatic support*,

$$\nabla\mathrm{p}_0 = -\rho_0\,\nabla\Phi_0\,. \tag{3.29}$$

Then, one can further rewrite (3.4), in particular by writing the term $\boldsymbol{\mathcal{P}}\boldsymbol{\xi} + \boldsymbol{\mathcal{G}}\boldsymbol{\xi}$ in terms of Eulerian perturbation δ_ρ^{E}. This is the Equation (3.43) of [11].

Proposition 2. *Given the hydrostatic equilibrium (3.29), the operator $\boldsymbol{\mathcal{P}}$ defined in (3.2) and the full gravity operator $\boldsymbol{\mathcal{G}}$ defined in (3.6) can be written in terms of the Eulerian perturbation quantities (3.11) as,*

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} + \boldsymbol{\mathcal{G}}\boldsymbol{\xi} = \nabla\delta_{\mathrm{p}}^{\mathrm{E}} + \delta_\rho^{\mathsf{E}}\nabla\Phi_0 + \rho_0\nabla\delta_\Phi^{\mathsf{E}}\,. \tag{3.30}$$

As a result of this, with Eulerian perturbations $\delta_\bullet^{\mathsf{E}}$ defined in (3.11), the Galbrun's equation (3.1) can

[3]The Euler's equation with adiabatic state for the background in Eulerian quantities, cf. [24, Eqn 1.14-1.17 p. 22] with exterior force (in this case gravity) $\mathbf{F}_0 := -\rho_0\nabla\Phi_0$ is

$$\begin{aligned}
\partial_t\rho_0 + \nabla\cdot(\rho\mathbf{v}) &= 0\,, && \text{Convervation of mass,}\\
\partial_r(\rho_0\,\mathrm{v}_0) + \nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v} + \mathrm{p}_0\,\mathbf{Id}) &= -\rho_0\nabla\Phi_0\,, && \text{Equation of motion,}\\
\gamma\,\mathrm{p}_0 &= \rho_0\,\mathrm{c}_0^2\,, && \text{Adiabatic equation of state.}
\end{aligned} \tag{3.27}$$

At equilibrium, and assuming no background flow ($\mathrm{v}_0 = 0$), then

$$\nabla\cdot(\rho\mathbf{v}) = 0\,,\qquad \nabla\mathrm{p}_0 = -\rho_0\nabla\Phi_0\,,\qquad \gamma\,\mathrm{p}_0 = \rho_0\,\mathrm{c}_0^2\,. \tag{3.28}$$

In particular, the equation of motion (3.27) takes the form of (3.29).

be written as

$$
\begin{aligned}
& -\rho_0\,\sigma^2\,\boldsymbol{\xi} \,+\, \nabla\delta_{\mathrm{p}}^{\mathsf{E}} \,+\, \delta_\rho^{\mathsf{E}}\,\nabla\Phi_0 \,+\, \rho_0\,\nabla\delta_\Phi^{\mathsf{E}} \,=\, \mathbf{f} \\
\Rightarrow\quad & -\rho_0\,\sigma^2\,\boldsymbol{\xi} \,+\, \nabla\delta_{\mathrm{p}}^{\mathsf{E}} \,-\, \delta_\rho^{\mathsf{E}}\,\frac{\nabla \mathrm{p}_0}{\rho_0} \,+\, \rho_0\,\nabla\delta_\Phi^{\mathsf{E}} \,=\, \mathbf{f}\,.
\end{aligned} \tag{3.31}
$$

Proof. Taking the gradient on both sides of (3.29)

$$
\begin{aligned}
& \nabla\nabla \mathrm{p}_0 \,=\, -\nabla(\rho_0\,\nabla\Phi_0) \\
& \Rightarrow \nabla\nabla \mathrm{p}_0 \,=\, -(\nabla\rho_0)\otimes(\nabla\Phi_0) \,-\, \rho_0\nabla\nabla\Phi_0 \\
& \Rightarrow \boldsymbol{\xi}\cdot\nabla\nabla \mathrm{p}_0 \,=\, -\boldsymbol{\xi}\cdot\big[(\nabla\rho_0)\otimes(\nabla\Phi_0)\big] \,-\, \rho_0\boldsymbol{\xi}\cdot\nabla\nabla\Phi_0\,.
\end{aligned} \tag{3.32}
$$

From (3.19), we have

$$
(\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \,=\, \boldsymbol{\xi}\cdot \underbrace{\nabla\nabla}_{\text{Hessian}}\, \mathrm{p}_0 \,=\, (\nabla\nabla \mathrm{p}_0)\cdot\boldsymbol{\xi}\,, \tag{3.33}
$$

and thus,

$$
(\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \,+\, \boldsymbol{\xi}\cdot\big[(\nabla\rho_0)\otimes(\nabla\Phi_0)\big] \,+\, (\rho_0\boldsymbol{\xi})\cdot\nabla\nabla\Phi_0 \,=\, 0\,. \tag{3.34}
$$

Next, the first and third terms in the defining expression (3.2) of $\boldsymbol{\mathcal{P}}$ combined can be expressed in terms of $\delta_{\mathrm{p}}^{\mathsf{E}}$,

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}\boldsymbol{\xi} \,&=\, -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] \,+\, (\nabla \mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) \,-\, \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \\
&=\, \nabla\delta_{\mathrm{p}}^{\mathrm{E}} \,+\, (\nabla \mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \\
&\overset{(3.25)}{=}\, \nabla\delta_{\mathrm{p}}^{\mathrm{E}} \,+\, (-\rho_0\,\nabla\Phi_0)(\nabla\cdot\boldsymbol{\xi}) \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \\
&\overset{(3.25)}{=}\, \nabla\delta_{\mathrm{p}}^{\mathrm{E}} \,+\, \delta_\rho^{\mathsf{E}}\nabla\Phi_0 \,+\, \boldsymbol{\xi}\cdot(\nabla\rho_0)\otimes(\nabla\Phi_0) \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0\,.
\end{aligned}
$$

For the last equality we have used

$$
\begin{aligned}
& (-\rho_0\,\nabla\Phi_0)(\nabla\cdot\boldsymbol{\xi}) \,=\, -(\rho_0\,\nabla\cdot\boldsymbol{\xi})\,(\nabla\Phi_0) \,=\, -\nabla\cdot(\rho_0\boldsymbol{\xi})\,(\nabla\Phi_0) \,+\, (\nabla\rho_0)\cdot\boldsymbol{\xi}(\nabla\Phi_0) \\
\Rightarrow\, & (-\rho_0\,\nabla\Phi_0)(\nabla\cdot\boldsymbol{\xi}) \,=\, \delta_\rho^{\mathsf{E}}\nabla\Phi_0 \,+\, \boldsymbol{\xi}\cdot(\nabla\rho_0)\otimes(\nabla\Phi_0)\,.
\end{aligned}
$$

On the other hand, in the notation of the Eulerian gravity perturbation(3.11c), the full gravity operator $\boldsymbol{\mathcal{G}}$ defined in (3.11) is written as

$$
\boldsymbol{\mathcal{G}}\boldsymbol{\xi} \,=\, \rho_0\nabla\delta_\Phi^{\mathsf{E}} \,+\, \rho_0\,(\boldsymbol{\xi}\cdot\nabla)\nabla(\Phi_0)\,,
$$

such that,

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}\boldsymbol{\xi} \,+\, \boldsymbol{\mathcal{G}}\boldsymbol{\xi} \,=\, & \nabla\delta_{\mathrm{p}}^{\mathrm{E}} \,+\, \delta_\rho^{\mathsf{E}}\nabla\Phi_0 \,+\, \rho_0\nabla\delta_\Phi^{\mathsf{E}} \\
& +\, \underbrace{\boldsymbol{\xi}\cdot(\nabla\rho_0)\otimes(\nabla\Phi_0) \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla \mathrm{p}_0 \,+\, \rho_0\,(\boldsymbol{\xi}\cdot\nabla)\nabla(\Phi_0)}_{=0 \;\text{ due to (3.34)}}\,.
\end{aligned} \tag{3.35}
$$

□

4 Galbrun equation in spherical symmetry

4.1 Decompositions of $\boldsymbol{\xi}$

We write the displacement $\boldsymbol{\xi}$ and source $\mathbf{f}$ in basis made up of vector spherical harmonics $\mathbf{P}_\ell^m$, $\mathbf{B}_\ell^m$ and $\mathbf{C}_\ell^m$ introduced in (2.20),

$$
\boldsymbol{\xi} = \boldsymbol{\xi}_r + \boldsymbol{\xi}_h\,, \tag{4.1}
$$

where

$$
\begin{aligned}
\boldsymbol{\xi}_r(r\,\widehat{x}) \,&=\, \pi_r\boldsymbol{\xi} \,=\, a_0^0(r)\,\mathbf{P}_0^0 \,+\, \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(r)\,\mathbf{P}_\ell^m(\widehat{x}) \quad ; \\
\boldsymbol{\xi}_h \,&=\, \pi_\theta\boldsymbol{\xi} + \pi_\phi\boldsymbol{\xi} \,=\, \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} b_\ell^m(r)\,\mathbf{B}_\ell^m(\widehat{x}) \,+\, \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} c_\ell^m(r)\,\mathbf{C}_\ell^m(\widehat{x}),\,.
\end{aligned}
$$

Remark 7. *To alleviate the notation in the harmonic expansion, we use the convention that*

$$\mathbf{B}_0^0 \,=\, \mathbf{C}_0^0 \,=\, 0\,. \tag{4.2}$$

and thus all of the coefficients tangential coefficients at $\ell = 0$ are zero,

$$b_0^0 \,=\, c_0^0 \,=\, 0\,. \tag{4.3}$$

△

We have (using convention in Remark 7),

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi} &= \frac{1}{r^2}\,\partial_r\,(r^2\pi_r\boldsymbol{\xi}) \,+\, \frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\Big[\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big]\,\mathrm{Y}_\ell^m \quad .\end{aligned} \tag{4.4}$$

Here, in the first term we have used $\pi_r\boldsymbol{\xi} \,=\, a_\ell^m\,\mathrm{Y}_\ell^m$. In the second term, we used (2.22) and (2.23).
We also have a similar decomposition for the source $\mathbf{f}$ (using convention in Remark 7)

$$\mathbf{f} \,=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m(r)\,\mathbf{P}_\ell^m(\widehat{x}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell^m(r)\,\mathbf{B}_\ell^m(\widehat{x}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} h_\ell^m(r)\,\mathbf{C}_\ell^m(\widehat{x})\,. \tag{4.5}$$

4.2 Decomposition of $\mathcal{P}$

We decompose

$$\boldsymbol{\mathcal{P}}\boldsymbol{\xi} \,=\, -\nabla\big[\gamma\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] \,+\, (\nabla\mathrm{p}_0)(\nabla\cdot\boldsymbol{\xi}) \,-\, \nabla[(\boldsymbol{\xi}\cdot\nabla)\mathrm{p}_0] \,+\, (\boldsymbol{\xi}\cdot\nabla)\nabla\mathrm{p}_0 \tag{4.6}$$

in the basis of vector spherical harmonics. We also follow the convention in Remark 7.

• Consider the second term in $\boldsymbol{\mathcal{P}}\boldsymbol{\xi}$. Using (4.4), the second term in $\boldsymbol{\mathcal{P}}\boldsymbol{\xi}$ can be written as

$$(\nabla\mathrm{p}_0)\,\nabla\cdot\boldsymbol{\xi} \,=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\mathrm{p}_0'\,\left[\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\,\mathrm{Y}_\ell^m\,\mathbf{e}_r \quad .$$

• Consider the first term in $\boldsymbol{\mathcal{P}}\boldsymbol{\xi}$

$$\begin{aligned}\nabla\big[\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\big] \,&=\, \nabla\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\gamma\mathrm{p}_0\Big[\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big]\mathrm{Y}_\ell^m \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\mathrm{Y}_\ell^m\,\nabla\left[\gamma\mathrm{p}_0\Big(\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big)\right] \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\gamma\mathrm{p}_0\Big[\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big]\,\nabla\mathrm{Y}_\ell^m \\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\partial_r\,\left[\gamma\mathrm{p}_0\Big(\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big)\right]\,\mathrm{Y}_\ell^m\,\mathbf{e}_r + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\frac{\gamma\mathrm{p}_0}{r}\,\Big(\frac{(r^2 a_\ell^m)'}{r^2} \,-\, \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big)\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \quad .\end{aligned}$$

• Consider the fourth term in $\boldsymbol{\mathcal{P}}\boldsymbol{\xi}$. This is the material derivative of vector $\nabla\mathrm{p}_0 = \mathrm{p}_0'\mathbf{e}_r$ in the direction $\boldsymbol{\xi}$.

$$\begin{aligned}(\boldsymbol{\xi}\cdot\nabla)\,\nabla\mathrm{p}_0 &= (\boldsymbol{\xi}\cdot\nabla)\,(\mathrm{p}_0'\,\mathbf{e}_r) \\ &= (\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0')\,\mathbf{e}_r + \frac{(\pi_\theta\boldsymbol{\xi})\,\mathrm{p}_0'}{r}\,\mathbf{e}_\theta + \frac{(\pi_\phi\boldsymbol{\xi})\,\mathrm{p}_0'}{r}\,\mathbf{e}_\phi \\ &= \mathrm{p}_0''\,(\boldsymbol{\xi}\cdot\mathbf{e}_r)\,\mathbf{e}_r + \frac{\mathrm{p}_0'}{r}\big[(\pi_\theta\boldsymbol{\xi})\,\mathbf{e}_\theta \,+\, (\pi_\phi\boldsymbol{\xi})\,\mathbf{e}_\phi\big]\,.\end{aligned}$$

Thus the fourth term in $\boldsymbol{\mathcal{P}}\boldsymbol{\xi}$ is

$$(\boldsymbol{\xi}\cdot\nabla)\,\nabla\mathrm{p}_0 \,=\, \mathrm{p}_0''\boldsymbol{\xi}_r \,+\, \frac{\mathrm{p}_0'}{r}\boldsymbol{\xi}_h\,. \tag{4.7}$$

- Consider the third term in $\boldsymbol{\mathcal{P}\xi}$. Since

$$(\boldsymbol{\xi}\cdot\nabla)\,\mathrm{p}_0 = \mathrm{p}_0'\boldsymbol{\xi}\cdot\mathbf{e}_r = \mathrm{p}_0'\boldsymbol{\xi}_r = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\mathrm{p}_0'\,a_\ell^m\,\mathrm{Y}_\ell^m\,, \tag{4.8}$$

we have

$$\begin{aligned}\nabla(\mathrm{p}_0'\,a_\ell^m\,\mathrm{Y}_\ell^m) &= (\nabla\mathrm{p}_0'a_\ell^m)\mathrm{Y}_\ell^m + \mathrm{p}_0'a_\ell^m\nabla\mathrm{Y}_\ell^m = (\nabla\mathrm{p}_0'a_\ell^m)\mathrm{Y}_\ell^m + \frac{\mathrm{p}_0'a_\ell^m}{r}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\\ &= (\mathrm{p}_0'a_\ell^m)'\,\mathrm{Y}_\ell^m\,\mathbf{e}_r + \frac{\mathrm{p}_0'a_\ell^m}{r}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,.\end{aligned}$$

As a result of this, the third term in $\boldsymbol{\mathcal{P}\xi}$ is written as

$$-\nabla[(\boldsymbol{\xi}\cdot\nabla)\,\mathrm{p}_0] \;=\; -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\mathrm{p}_0'a_\ell^m)'\,\mathrm{Y}_\ell^m\,\mathbf{e}_r - \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\mathrm{p}_0'a_\ell^m}{r}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,. \tag{4.9}$$

By assembling the terms, we obtain

$$\begin{aligned}\boldsymbol{\mathcal{P}\,\xi} = &-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\partial_r\left[\gamma\mathrm{p}_0\Big(\frac{(r^2a_\ell^m)'}{r^2} \;-\; \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big)\right]\mathrm{Y}_\ell^m\,\mathbf{e}_r\\ &-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\frac{\gamma\mathrm{p}_0}{r}\,\Big(\frac{(r^2a_\ell^m)'}{r^2} \;-\; \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big)\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\\ &+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\mathrm{p}_0'\,\left[\frac{(r^2a_\ell^m)'}{r^2} \;-\; \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\;\mathrm{Y}_\ell^m\,\mathbf{e}_r\\ &+\mathrm{p}_0''\boldsymbol{\xi}_r \;+\; \frac{\mathrm{p}_0'}{r}\boldsymbol{\xi}_h\\ &-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\mathrm{p}_0'a_\ell^m)'\,\mathrm{Y}_\ell^m\,\mathbf{e}_r - \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\mathrm{p}_0'a_\ell^m}{r}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \quad .\end{aligned} \tag{4.10}$$

4.3 Gravitational contribution

Following (4.7),

$$\boldsymbol{\mathcal{G}\xi} \;=\; \rho_0\,(\boldsymbol{\xi}\cdot\nabla)\nabla(\Phi_0) \;=\; \rho_0\,\Phi_0''\,\boldsymbol{\xi}_r \;+\; \rho_0\,\frac{\Phi_0'}{r}\,\boldsymbol{\xi}_h\,, \tag{4.11}$$

we further obtain explicit the expressions of Φ_0'' and Φ_0' by using equation

$$\Delta\Phi_0 \;=\; 4\,\pi\,G\,\rho_0. \tag{4.12}$$

Under the radial assumption, the left-hand-side reduces to

$$\Delta\Phi_0 \;=\; \Phi_0'' \;+\; \frac{2}{r}\Phi_0' \;=\; \frac{1}{r^2}(r^2\,\Phi')'\,. \tag{4.13}$$

Thus (1.2) is written as

$$\frac{1}{r^2}(r^2\,\Phi_0')' \;=\; 4\,\pi\,G\,\rho_0\,. \tag{4.14}$$

From (4.14), for both the atmosphere and interior of the Sun, we have

$$\boxed{\Phi_0' = \frac{4\pi G}{r^2}\int_0^r s^2\rho_0(s)\,ds\,,} \tag{4.15}$$

and

$$\boxed{\Phi_0'' \;=\; 4\pi G\rho_0 \;-\; \frac{2}{r}\Phi_0'\,.} \tag{4.16}$$

4.3.1 In the interior of the Sun

We recall from (3.14)

$$\mathrm{p}_0' - \rho_0\,\Phi_0' = 0 \quad , \quad r \leq r_a \qquad \Rightarrow \qquad \mathrm{p}_0'' - \rho_0'\,\Phi_0' - \rho_0\,\Phi_0'' = 0\,. \tag{4.17}$$

In addition to expression (4.15) and (4.16), in the interior of the Sun, the first and second-order radial derivatives of Φ_0 are also written as,

$$\boxed{\Phi_0' = \frac{\mathrm{p}_0'}{\rho_0} \quad , \quad \Phi_0'' = \frac{\mathrm{p}_0''}{\rho_0} - \frac{\rho_0'}{\rho_0}\Phi_0' = \frac{\mathrm{p}_0''}{\rho_0} - \frac{\rho_0'}{\rho_0}\frac{\mathrm{p}_0'}{\rho_0}\,.} \tag{4.18}$$

4.3.2 In the atmosphere with model `AtmoCAI`

We have assumed that ρ_0 decreases exponentially for $r \geq r_a$, i.e.

$$\rho_0(r) = \rho_0(r_a)\,e^{-\alpha_{\rho_0}\,(r-r_a)} \quad , \quad r \geq r_a \quad , \quad \text{constant } \alpha_{\rho_0} > 0\,. \tag{4.19}$$

Since

$$\partial_r \left(-\frac{e^{-\alpha_{\rho_0} s}}{(\alpha_{\rho_0})^3} \left((\alpha_{\rho_0}\,s)^2 + 2\,s\alpha_{\rho_0} + 2 \right) \right) = s^2\,e^{\alpha_{\rho_0}\,s}\,,$$

for $r \geq r_a$, we have

$$\begin{aligned}
\Phi_0'(r) &= \frac{G}{r^2}\int_0^r s^2\rho_0(s)\,ds \\
&= \frac{4\pi G}{r^2}\int_0^{r_a} s^2\rho_0(s)\,ds + \frac{4\pi G}{r^2}\rho_0(r_a)\,e^{\alpha_{\rho_0} r_a}\int_{r_a}^r s^2\,e^{-\alpha_{\rho_0}\,r}\,ds \\
&= \frac{4\pi G}{r^2}\int_0^{r_a} s^2\rho_0(s)\,ds + \frac{4\pi G}{r^2}\rho_0(r_a)\left(\frac{e^{-\alpha_{\rho_0}(s-r_a)}}{(\alpha_{\rho_0})^3}\left((\alpha_{\rho_0}\,s)^2 + 2\,s\alpha_{\rho_0} + 2\right)\Big|_r^{r_a}\right).
\end{aligned}$$

This means that Φ_0' is a sum of a multiple of r^{-2} and a term that is exponentially decaying at ∞.

Lemma 1. *Under the assumption (4.19) i.e. ρ_0 decreases exponentially in $r \geq r_a$, with Φ_0 defined by (1.2) and $r \geq r_a$, we have*

$$\Phi_0'(r) = \frac{G}{r^2}\mathfrak{m} - 4\,\pi\,G\,\rho_0(r_a)\frac{e^{-\alpha_{\rho_0}(r-r_a)}}{r^2}\,\frac{(\alpha_{\rho_0}\,r)^2 + 2\,r\alpha_{\rho_0} + 2}{(\alpha_{\rho_0})^3}\,, \tag{4.20}$$

where the constant $\mathfrak{m}$ (depending on r_a and α_{ρ_0}),

$$\mathfrak{m} := 4\pi\int_0^{r_a} s^2\rho_0(s)\,ds + 4\pi\,\rho_0(r_a)\frac{(\alpha_{\rho_0}\,r_a)^2 + 2\,r_a\alpha_{\rho_0} + 2}{(\alpha_{\rho_0})^3}\,. \tag{4.21}$$

Remark 8. *We note that the first term in the definition (4.21) of $\mathfrak{m}$*

$$4\pi\int_0^{r_a} s^2\rho_0(s)\,ds$$

is the mass of the Sun until the beginning of the atmosphere, while the second term which can be written as

$$4\pi\,\rho_0(r_a)\frac{(\alpha_{\rho_0}\,r_a)^2 + 2\,r_a\alpha_{\rho_0} + 2}{(\alpha_{\rho_0})^3} = \int_{r_a}^{\infty} s^2\rho_0(s)\,ds\,,$$

can be considered as the exterior mass of the Sun (to infinity). In this way, the constant $\mathfrak{m}$ is the mass of the 'infinite' Sun. △

4.4 System of ODEs

We employ the convention in Remark 7. Putting together result of (4.10) for $\boldsymbol{\mathcal{P}\xi}$ and (4.11) for $\boldsymbol{\mathcal{G}\xi}$, we obtain

$$\begin{aligned}
&\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m(r)\,\mathrm{Y}_\ell^m\,\mathbf{e}_r + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{g_\ell^m(r)}{\sqrt{\ell(\ell+1)}}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m + \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{h_\ell^m(r)}{\sqrt{\ell(\ell+1)}}(-\mathbf{e}_r\times\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m)\\
&\qquad = -\rho_0\,\sigma^2\,\boldsymbol{\xi}\\
&-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\partial_r\left[\gamma\mathrm{p}_0\Big(\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big)\right]\mathrm{Y}_\ell^m\,\mathbf{e}_r - \sum_{\ell=0}^{\infty}\sum_{m=-\ell}\frac{\gamma\mathrm{p}_0}{r}\Big(\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big)\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\mathrm{p}_0'\left[\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\mathrm{Y}_\ell^m\,\mathbf{e}_r\\
&+(\mathrm{p}_0'' + \rho_0\,\Phi_0'')\,\boldsymbol{\xi}_r + \frac{\mathrm{p}_0' + \rho_0\,\Phi_0'}{r}\,\boldsymbol{\xi}_h\\
&-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\mathrm{p}_0'a_\ell^m)'\,\mathrm{Y}_\ell^m\,\mathbf{e}_r - \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\mathrm{p}_0'a_\ell^m}{r}\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \quad .
\end{aligned}\tag{4.22}$$

In particular, the coefficient of $\mathrm{Y}_\ell^m\mathbf{e}_r$ is

$$\begin{aligned}
&-\rho_0\,\sigma^2\,\boldsymbol{\xi}_r - \partial_r\left[\gamma\mathrm{p}_0\Big(\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big)\right] + \mathrm{p}_0'\left[\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\\
&\qquad + \rho_0\,\Phi_0''\,a_\ell^m + \underbrace{\mathrm{p}_0''a_\ell^m - (\mathrm{p}_0'a_\ell^m)'}_{-\mathrm{p}_0'\,\partial_r a_\ell^m}\,.
\end{aligned}$$

ODE in the radial direction obtained as coefficients of $\mathrm{Y}_\ell^m\,\mathbf{e}_r$ in (4.22),

$$\begin{aligned}
&(-\sigma^2\,\rho_0 + \rho_0\,\Phi_0'')\,a_\ell^m - \partial_r\left[\gamma\mathrm{p}_0\Big(\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\Big)\right]\\
&\qquad + \mathrm{p}_0'\left[\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right] - \mathrm{p}_0'\,\partial_r a_\ell^m = f_\ell^m\,.
\end{aligned}\tag{4.23}$$

ODE in the tangential direction obtained as coefficients of $\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m$ in (4.22),

$$\begin{aligned}
&-\sigma^2\rho_0\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\\
&-\frac{\gamma\mathrm{p}_0}{r}\Big(\frac{(r^2a_\ell^m)'}{r^2} - \sqrt{\ell(\ell+1)}\frac{b_\ell^m}{r}\Big) + \frac{\mathrm{p}_0' + \rho_0\,\Phi_0'}{r}\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} - \frac{\mathrm{p}_0'\,a_\ell^m}{r} = \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}}\,,
\end{aligned}\tag{4.24}$$

and coefficients of $-\mathbf{e}_r\times\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m$

$$\Big(-\sigma^2\rho_0 + \frac{\mathrm{p}_0' + \rho_0\Phi_0'}{r}\Big)\frac{c_\ell^m}{\sqrt{\ell(\ell+1)}} = \frac{h_\ell^m}{\sqrt{\ell(\ell+1)}}\,.\tag{4.25}$$

Equation (4.25) implies that

$$\begin{aligned}
c_\ell^m &= \frac{h_\ell^m(r)}{-\sigma^2\rho_0 + \frac{\mathrm{p}_0'+\rho_0\Phi_0'}{r}}\quad, \quad r \geq r_a\\
c_\ell^m &= \frac{h_\ell^m(r)}{-\sigma^2\,\rho_0(r)} \qquad r < r_a\,,
\end{aligned}\tag{4.26}$$

For the equation in the interior, we have used the relation (3.14), which gives that $p_0' + \rho_0\Phi_0' = 0$ for $r \leq r_a$. See also Remark 9 for the equation at $\ell = 0$.

For the rest of the work, we will focus on a_ℓ^m and b_ℓ^m. Using the identities

$$\begin{aligned}
\frac{(r^2 a_l^{m})'}{r^2} &= \frac{2}{r} a_l^{m} + \partial_r a_l^{m}\,;\\
\partial_r \left[\gamma p_0 \frac{(r^2 a_l^{m})'}{r^2}\right] &= \partial_r \left[\frac{\gamma p_0}{r^2}(2 r a_l^{m} + r^2 \partial_r a_l^{m})\right] = \partial_r \left[\gamma p_0 (\frac{2}{r} a_l^{m} + \partial_r a_l^{m})\right]\\
&= (\gamma p_0)' \left(\frac{2}{r} a_l^{m} + \partial_r a_l^{m}\right) + \gamma p_0 \big(-\frac{2}{r^2} a_l^{m} + \frac{2}{r}\partial_r a_l^{m} + \partial_r^2 a_l^{m}\big)\\
&= \left[(\gamma p_0)' \frac{2}{r} - \gamma p_0 \frac{2}{r^2}\right] a_l^{m} + \left[(\gamma p_0)' + \frac{2}{r}\gamma p_0\right]\partial_r a_l^{m} + \gamma p_0\, \partial_r^2 a_l^{m}\,,
\end{aligned}$$

we rewrite (4.23) and (4.24) in matrix form,

$$\begin{aligned}
0 = &\begin{pmatrix} -\gamma p_0 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}\\
&+ \begin{pmatrix} -(\gamma p_0)' - \dfrac{2\gamma p_0}{r} & \dfrac{\gamma p_0}{r}\ell(\ell+1) \\ -\dfrac{\gamma p_0}{r} & 0 \end{pmatrix} \partial_r \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}\\
&+ \begin{pmatrix} -\sigma^2 \rho_0 + \rho_0 \Phi_0'' - \dfrac{2(\gamma p_0)'}{r} + \dfrac{2\gamma_0 p_0}{r^2} + \dfrac{2 p_0'}{r} & \ell(\ell+1)\left(\dfrac{-p_0' + (\gamma p_0)'}{r} - \dfrac{\gamma p_0}{r^2}\right) \\ -\dfrac{2\gamma p_0}{r^2} - \dfrac{p_0'}{r} & -\sigma^2 \rho_0 + \dfrac{\gamma p_0}{r^2}\ell(\ell+1) + \dfrac{p_0' + \rho_0 \Phi_0'}{r} \end{pmatrix} \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}.
\end{aligned} \tag{4.27}$$

Dividing both sides of (4.27) by $\gamma\, p_0$, we obtain

$$\boxed{\begin{aligned}
\frac{1}{\gamma\, p_0}\begin{pmatrix} f_\ell^m \\ \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} = &\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} + \begin{pmatrix} -\dfrac{(\gamma p_0)'}{\gamma p_0} - \dfrac{2}{r} & \dfrac{\ell(\ell+1)}{r} \\ -\dfrac{1}{r} & 0 \end{pmatrix} \partial_r \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}\\
+ &\begin{pmatrix} -\dfrac{\sigma^2\rho_0}{\gamma p_0} + \dfrac{\rho_0 \Phi_0''}{\gamma p_0} - \dfrac{2}{r}\dfrac{(\gamma p_0)'}{\gamma p_0} + \dfrac{2}{r^2} + \dfrac{2}{r\gamma}\dfrac{p_0'}{p_0} & \ell(\ell+1)\left(\dfrac{1}{r}\left[-\dfrac{1}{\gamma}\dfrac{p_0'}{p_0} + \dfrac{(\gamma p_0)'}{\gamma p_0}\right] - \dfrac{1}{r^2}\right) \\ -\dfrac{2}{r^2} - \dfrac{1}{r\gamma}\dfrac{p_0'}{p_0} & -\sigma^2 \dfrac{\rho_0}{\gamma p_0} + \dfrac{\ell(\ell+1)}{r^2} + \dfrac{1}{r\,\gamma}\left(\dfrac{p_0'}{p_0} + \rho_0 \dfrac{\Phi_0'}{p_0}\right) \end{pmatrix}\\
&\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}.
\end{aligned}} \tag{4.28}$$

For convenience, we define matrix B and C so that (4.28) is written as

$$\frac{1}{\gamma\, p_0}\begin{pmatrix} f_\ell^m \\ \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} + B\,\partial_r \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} + C \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}. \tag{4.29}$$

Remark 9 (Radial equation at $\ell = 0$)**.** *We note that for $\ell = 0$ (recall the convention in Remark 7),*

$$b_0^0 = c_0^0 = g_0^0 = h_0^0 = 0\,. \tag{4.30}$$

Equation (4.23) is only in terms of a_0^0,

$$\boxed{\big(-\sigma^2 \rho_0 + \rho_0 \Phi_0''\big)\, a_0^0 - \partial_r \left(\frac{\gamma p_0}{r^2}\,\partial_r (r^2 a_0^0)\right) + \frac{p_0'}{r^2}\,\partial_r (r^2 a_0^0) - p_0'\,\partial_r a_0^0 = f_0^0\,.} \tag{4.31}$$

Following the same algebraic steps as above, or equivalently taking the first row of (4.28) (or equivalently (4.29)) at $\ell = 0$, we arrive at

$$\boxed{-\partial_r^2\, a_0^0 \,+\, \left(2\alpha_{\mathrm{p}_0} \,-\, \frac{2}{r}\right)\,\partial_r\, a_0^0 \,+\, C_{11}\, a_0^0 \,=\, 0\,.} \tag{4.32}$$

$\triangle$

We next rewrite (4.29) in terms of the various scale height functions introduced in (2.6),

$$\alpha\,(=\,\alpha_{\rho_0}) \,=\, -\frac{\rho_0'}{\rho_0} \quad,\quad \alpha_{\mathrm{c}_0} \,:=\, -\frac{\mathrm{c}_0'}{\mathrm{c}_0} \quad,\quad \alpha_\gamma \,:=\, -\frac{\gamma'}{\gamma}\,,$$

4.4.1 In the interior of the Sun

Using hydrostatic equilibrium (3.14), the last term in the component C_{22} vanishes. Additionally, in using the adiabatic equation of state, one can replace $\gamma\mathrm{p}_0$ by $\mathrm{c}_0^2\rho_0$, or vice versa. With these two ingredients, one can rewrite (4.28) as

$$\boxed{\begin{aligned}&\frac{1}{\gamma\,\mathrm{p}_0}\begin{pmatrix} f_\ell^m \\ \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix} \,=\, \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\partial_r^2\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix} \,+\, \begin{pmatrix} -\frac{(\gamma\mathrm{p}_0)'}{\gamma\mathrm{p}_0} \,-\, \frac{2}{r} & \frac{\ell(\ell+1)}{r} \\ -\frac{1}{r} & 0\end{pmatrix}\partial_r\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix}\\ &+\begin{pmatrix} -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\Phi_0''}{\mathrm{c}_0^2} - \frac{2}{r}\frac{(\gamma\mathrm{p}_0)'}{\gamma\mathrm{p}_0} + \frac{2}{r^2} + \frac{2}{r\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0} & \ell(\ell+1)\left(\frac{1}{r}\left[-\frac{1}{\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0} + \frac{(\gamma\mathrm{p}_0)'}{\gamma\mathrm{p}_0}\right] - \frac{1}{r^2}\right) \\ -\frac{2}{r^2} \,-\, \frac{1}{r\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0} & -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\ell(\ell+1)}{r^2}\end{pmatrix}\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix}.\end{aligned}} \tag{4.33}$$

In terms of the scale height functions (2.6), system (4.33) takes the following form. We note that the adiabatic exponent γ *is not* constant here.

In the interior of the Sun, for $r \leq r_a$, the system (4.29) has the following explicit form

$$\begin{aligned}\frac{1}{\gamma\,\mathrm{p}_0}\begin{pmatrix} f_\ell^m \\ \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix} \,&=\, \begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\partial_r^2\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix}\\ &+\begin{pmatrix} 2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \frac{2}{r} & \frac{\ell(\ell+1)}{r} \\ -\frac{1}{r} & 0\end{pmatrix}\partial_r\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix} \,+C\begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\end{pmatrix},\end{aligned} \tag{4.34}$$

where

$$C_{11} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,+\, 2\frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,+\, \frac{2}{r^2} \,-\, 2\frac{\alpha_{\mathrm{p}_0}}{r\gamma}\,; \tag{4.35a}$$

$$C_{12} \,=\, \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{r\,\gamma} \,-\, \frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,-\, \frac{1}{r^2}\right)\,; \tag{4.35b}$$

$$C_{21} \,=\, -\frac{2}{r^2} \,+\, \frac{1}{r\gamma}\,\alpha_{\mathrm{p}_0}\,; \tag{4.35c}$$

$$C_{22} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\ell(\ell+1)}{r^2}\,. \tag{4.35d}$$

4.4.2 In the atmosphere with model `AtmoCAI`

In the atmosphere, γ and c_0 are constant. With the assumption of ideal atmospheric pressure, the scale height associated the background density α is constant, while the other scale heights are zeros. We have

$$(\mathrm{c}_0^2\,\rho_0)' \,=\, \mathrm{c}_0^2\,\rho_0'\,.$$

From the equation of state (3.13), $\dfrac{\rho_0}{\gamma \mathrm{p}_0} = \dfrac{\omega^2}{\mathrm{c}_0^2}$, we have

$$\frac{\mathrm{p}_0'}{\mathrm{p}_0} = \frac{(\gamma\,\mathrm{p}_0)'}{\gamma\,\mathrm{p}_0} = \frac{\mathrm{c}_0^2\,\rho_0'}{\mathrm{c}_0^2\,\rho_0} = \frac{\rho_0'}{\rho_0} = -\alpha\,. \tag{4.36}$$

This means that p_0 decays exponentially at the same rate as ρ_0 for $r \geq r_a$. To treat the term with Φ_0'' we use (4.16),

$$\frac{\rho_0 \Phi_0''}{\gamma \mathrm{p}_0} = \frac{\rho_0\,(4\pi G \rho_0 \,-\, \frac{2}{r}\Phi_0')}{\gamma \mathrm{p}_0} = \frac{\rho_0\, 4\pi\, G\, \rho_0}{\mathrm{c}_0^2\,\rho_0} - \frac{2}{r\mathrm{c}_0^2}\,\Phi_0' = \frac{4\pi G}{\mathrm{c}_0^2}\rho_0 - \frac{2}{\mathrm{c}_0^2}\,\frac{\Phi_0'}{r}\,. \tag{4.37}$$

In the atmosphere,

$$0 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} + \begin{pmatrix} \alpha \,-\, \frac{2}{r} & \frac{\ell(\ell+1)}{r} \\ -\frac{1}{r} & 0 \end{pmatrix} \partial_r \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} + C \begin{pmatrix} a_\ell^m \\ \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix}, \tag{4.38}$$

where

$$C_{11} = -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\Phi_0''}{\mathrm{c}_0^2} + \frac{2}{r}\alpha + \frac{2}{r^2} - \frac{2}{r}\frac{\alpha}{\gamma} \tag{4.39a}$$

$$= -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{2}{r}\left(\alpha - \frac{\alpha}{\gamma}\right) + \frac{2}{r^2} - \frac{2}{\mathrm{c}_0^2}\frac{\Phi_0'}{r} + \frac{4\pi G}{\mathrm{c}_0^2}\rho_0\,; \tag{4.39b}$$

$$C_{12} = \ell(\ell+1)\left(\frac{1}{r}\left(-\alpha + \frac{\alpha}{\gamma}\right) - \frac{1}{r^2}\right)\,; \tag{4.39c}$$

$$C_{21} = -\frac{2}{r^2} + \frac{1}{r}\frac{\alpha}{\gamma}\,; \tag{4.39d}$$

$$C_{22} = -\frac{\sigma^2}{\mathrm{c}_0^2} - \frac{\alpha}{\gamma}\frac{1}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\Phi_0'}{\mathrm{c}_0^2}\frac{1}{r}\,. \tag{4.39e}$$

4.5 Decoupled system

After the simplification in Subsection 4.4.1 and Subsection 4.4.2, the system of equation (with unknowns a_ℓ^m and b_ℓ^m)(4.38) for the atmosphere, and (4.34) for the interior can be unified as

$$\begin{aligned}\frac{1}{\gamma \mathrm{p}_0} \begin{pmatrix} f_\ell^m \\ \frac{g_\ell^m}{\sqrt{\ell(\ell+1)}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \partial_r^2 \begin{pmatrix} a_\ell^m \\ \tilde{b}_\ell^m \end{pmatrix} + \begin{pmatrix} 2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \frac{2}{r} & \frac{\ell(\ell+1)}{r} \\ -\frac{1}{r} & 0 \end{pmatrix} \partial_r \begin{pmatrix} a_\ell^m \\ \tilde{b}_\ell^m \end{pmatrix} \\ + \begin{pmatrix} C_{11} & C_{12} \\ -\frac{2}{r^2} + \frac{2\,\alpha_{\mathrm{c}_0} + \alpha -\, \alpha_\gamma}{r\gamma} & C_{22} \end{pmatrix} \begin{pmatrix} a_\ell^m \\ \tilde{b}_\ell^m \end{pmatrix},\end{aligned} \tag{4.40}$$

with C_{11}, C_{12} and C_{22} given by (4.35) for the interior and by (4.39) for the atmosphere. We note that in the atmosphere $r \geq r_a$,

$$\alpha_{\mathrm{c}_0} = \alpha_\gamma = 0 \quad , \quad r \geq r_a\,. \tag{4.41}$$

Proposition 3. *With the scale height quantities $\alpha_\bullet$ defined in (2.6), the radial coefficient $a(r) = a_\ell^m(r)$ solves the ODE*

$$\big(\,\hat{q}_\ell(r)\,\partial_r^2 + q_\ell(r)\,\partial_r + \tilde{q}_\ell(r)\,\big)\,a_\ell^m = \mathfrak{f}_\ell^m(r)\,. \tag{4.42}$$

where the right-hand side is given as

$$\mathfrak{f}_\ell^m = -\frac{C_{12}}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}} - \frac{\ell(\ell+1)}{r}\partial_r\left(\frac{1}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}}\right) + \frac{f_\ell^m}{\gamma\,\mathrm{p}_0}\,, \tag{4.43}$$

and coefficients,

$$\hat{q}_\ell(r) \,=\, -1 + \frac{\ell(\ell+1)}{r^2\, C_{22}}\,; \tag{4.44a}$$

$$q_\ell(r) \,=\, \alpha_{\gamma \mathrm{p}_0} \,-\, \frac{2}{r} \,+\, \frac{C_{12}}{r\, C_{22}} + \frac{\ell(\ell+1)}{r}\left(\frac{1}{r\, C_{22}}\right)' + \frac{\ell(\ell+1)}{r}\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\,; \tag{4.44b}$$

$$\tilde{q}_\ell(r) \,=\, C_{11} \,+\, \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{C_{12}}{r\, C_{22}} + \frac{\ell(\ell+1)}{r}\left[\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\right]'\,; \tag{4.44c}$$

$$\tag{4.44d}$$

The tangential coefficent b_ℓ^m is obtained by

$$\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \,=\, \frac{1}{r}\frac{1}{C_{22}}\partial_r a_\ell^m \,+\, \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}} a_\ell^m \,+\, \frac{1}{C_{22}}\frac{g_\ell^m}{\gamma\, \mathrm{p}_0\, \sqrt{\ell(\ell+1)}}\,. \tag{4.45}$$

Remark 10 (Radial equation at $\ell = 0$)**.** *As a continuation of Remark 9, the ODE (4.42) is consistent at $\ell = 0$, i.e. evaluated at $\ell = 0$, it gives back (4.32). Note that $C_{12}|_{\ell=0} = 0$ and $C_{22}|_{\ell=0} = -\frac{\sigma^2}{\mathrm{c}_0^2}$.* △

Remark 11 (In the atmo)**.** *Another equivalent form of the above equations can be obtained by writing $\alpha_{\mathrm{p}_0} = 2\,\alpha_{\mathrm{c}_0} + \alpha - \alpha_\gamma$. For convenience, in using (4.41), we note here the form taken by expression (4.44b) and (4.44c) in the atmosphere, i.e. for $r \geq r_a$,*

$$\begin{aligned} q_\ell(r) \,&=\, \alpha \,-\, \frac{2}{r} \,+\, \frac{1}{r}\frac{C_{12}}{C_{22}} \,+\, \frac{\ell(\ell+1)}{r}\left[\left(\frac{1}{r\, C_{22}}\right)' + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\right]; \\ \tilde{q}_\ell(r) \,&=\, C_{11} \,+\, \frac{\ell(\ell+1)}{r}\left[\left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\right]' + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{C_{12}}{rC_{22}}\,. \end{aligned} \tag{4.46}$$

The tangential coefficent b_ℓ^m is obtained by

$$b_\ell^m \,=\, \frac{\sqrt{\ell(\ell+1)}}{r}\frac{1}{C_{22}}\partial_r a \,+\, \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{\sqrt{\ell(\ell+1)}}{rC_{22}}\, a\,. \tag{4.47}$$

△

Proof. We use the second equation of (4.40) to eliminate b_ℓ^m from a_ℓ^m.

$$\begin{aligned} -\frac{1}{r}\partial_r a_\ell^m \,+\, \left(-\frac{2}{r^2} \,+\, \frac{2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \alpha_\gamma}{r\gamma}\right) a_\ell^m \,+\, C_{22}\,\tilde{b}_\ell^m \,&=\, \frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}} \\ \Rightarrow\;\; \tilde{b} \,=\, \frac{1}{r}\frac{1}{C_{22}}\partial_r a_\ell^m \,+\, \left(\frac{2}{r} \,-\, \frac{2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \alpha_\gamma}{\gamma}\right)\frac{1}{rC_{22}} a_\ell^m \,&+\, \frac{1}{C_{22}}\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}}\,. \end{aligned}$$

Thus

$$C_{12}\,\tilde{b} \,=\, \frac{1}{r}\frac{C_{12}}{C_{22}}\partial_r a_\ell^m \,+\, \left(\frac{2}{r} \,-\, \frac{2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \alpha_\gamma}{\gamma}\right)\frac{C_{12}}{r\,C_{22}} a_\ell^m \,+\, \frac{C_{12}}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}} \tag{4.48}$$

and

$$\begin{aligned} \partial_r\,\tilde{b} \,=\, &\left(\frac{1}{r\,C_{22}}\right)'\partial_r a \,+\, \left(\frac{1}{r\,C_{22}}\right)\partial_r^2 a \,+\, \left[\left(\frac{2}{r} \,-\, \frac{2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \alpha_\gamma}{\gamma}\right)\frac{1}{rC_{22}}\right]' a \\ &+\, \left(\frac{2}{r} \,-\, \frac{2\,\alpha_{\mathrm{c}_0} \,+\, \alpha \,-\, \alpha_\gamma}{\gamma}\right)\frac{1}{rC_{22}}\,\partial_r a \,+\, \partial_r\left(\frac{1}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}}\right). \end{aligned} \tag{4.49}$$

□

Explicit expressions of the coefficients in the interior The derivation is given in Appendix A. We introduce the notation,

$$k_0 \;=\; \frac{\sigma}{\mathrm{c}_0}\,. \tag{4.50}$$

Lemma 2. *For $\ell > 0$, for $r \leq r_a$,*

$$\frac{C_{22}'}{C_{22}} \;=\; -\frac{2}{r} \,+\, r^2\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, r\,\frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\,. \tag{4.51}$$

Proposition 4. *For $r \leq r_a$, the coefficients of the ODE (7.12) are given by the following expressions.*

1. *The coefficient of the first order term has the form,*

$$r^2 C_{22}\hat{q} \;=\; k_0^2\,r^2\,. \tag{4.52}$$

$$r\,C_{22}\,q(r) \;=\; -\alpha_{\gamma\mathrm{p}_0}\frac{\sigma^2}{\mathrm{c}_0^2}\,r \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2} \,-\, \ell(\ell+1)\,\frac{r\left(2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\right) \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}}{\frac{\sigma^2}{\mathrm{c}_0^2}\,r^2 \,-\, \ell(\ell+1)}\,, \tag{4.53}$$

or equivalently

$$\frac{r\,C_{22}\,q(r)}{k_0^2} \;=\; -\alpha_{\gamma\mathrm{p}_0}\,r \,+\, 2 \,-\, \frac{\ell(\ell+1)}{k_0^2}\,\frac{2\,\alpha_{\mathrm{c}_0}\,r \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\,\frac{1}{k_0^2}\,r \,+\, 2}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,. \tag{4.54}$$

2. *The coefficient of the 0th-term is given by,*

$$\begin{aligned}
r^2\,C_{22}\,\tilde{q}(r) \;=\; &-\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\frac{(-\sigma^2 \,+\, \Phi_0'')}{\mathrm{c}_0^2}\,r^2 \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0} r}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0}\,r \,-\, 1\right)\\
&-\,\ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \ell(\ell+1)\frac{\Phi_0'}{\mathrm{c}_0^2}\left(-\frac{\Phi_0'}{\mathrm{c}_0^2} \,+\, \alpha_{\rho_0}\right)\\
&-\,\ell(\ell+1)\left(2 \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right)\frac{r\left(2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\right) \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}}{\frac{\sigma^2}{\mathrm{c}_0^2}\,r^2 \,-\, \ell(\ell+1)}\,.
\end{aligned} \tag{4.55}$$

A form entirely in terms of ρ_0, c_0 and Φ_0 , and k_0 is given as

$$\begin{aligned}
\frac{r^2\,C_{22}\,\tilde{q}(r)}{k_0^2} \;=\; &\left(k_0^2 \,-\, \frac{\Phi_0''}{\mathrm{c}_0^2}\right)\,r^2 \,+\, 2\,r\left(\frac{\Phi_0'}{\mathrm{c}_0^2} \,-\, \alpha_{\rho_0} - 2\alpha_{\mathrm{c}_0}\right) \,-\, 2 - \,\ell(\ell+1)\\
&+\,\frac{\ell(\ell+1)}{k_0^2}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\left(\alpha_{\rho_0} \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2}\right)\\
&-\,\frac{\ell(\ell+1)}{k_0^2}\left(2 \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2}\,r\right)\frac{2\,\alpha_{\mathrm{c}_0}\,r \,+\, 2 \,+\, \mathrm{i}\,\frac{\omega}{k_0^2}\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}r}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,.
\end{aligned} \tag{4.56}$$

Explicit expressions of the coefficient in the atmosphere The derivation is given in Appendix B. We introduce the notation

$$\boxed{E_{\mathrm{he}} \;:=\; -\,\frac{\alpha_{\rho_0}}{\gamma} \,+\, \frac{\Phi_0'}{\mathrm{c}_0^2}\,.} \tag{4.57}$$

Proposition 5. *In $r \geq r_a$, the coefficients of the ODE (7.12) are given by,*

$$r^2\, C_{22}\, \hat{q}(r) \;=\; r(k_0^2\, r \;-\; E_{he})\,, \tag{4.58}$$

$$\begin{aligned} r^2\, C_{22}\, q(r) &= (\alpha_{\rho_0} r \;-\; 2)\,(-k_0^2\, r \;+\; E_{he}) \\ &+ \ell(\ell+1)\frac{(k_0^2)'\, r^2 \;+\; 2\, k_0^2\, r \;-\; (E_{he} \;+\; r\frac{\Phi_0''}{\mathrm{c}_0^2})}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{he}}\,, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} & r^2\, C_{22}\, \tilde{q}(r) \\ &= \; (k_0^2\, r^2 \;-\; rE_{he})\left(k_0^2 \;-\; \frac{\Phi_0''}{\mathrm{c}_0^2}\right) \;+\; 2\,(k_0^2 r^2 - rE_{he})\left(-\frac{\alpha_{\rho_0}}{r} \;+\; \frac{\alpha_{\rho_0}}{r\,\gamma} \;-\; \frac{1}{r^2}\right) \\ &+\; \ell(\ell+1)\left(-k_0^2 + \frac{\Phi_0''}{\mathrm{c}_0^2}\right) \;-\; \ell(\ell+1)\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \;-\; \alpha_{\rho_0}\right) \\ &+ \ell(\ell+1)\left(\frac{2}{r} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\, r^2 \;+\; 2\, k_0^2\, r \;-\; \left(E_{he} \;+\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{he}}\,. \end{aligned} \tag{4.60}$$

4.6 Reduction to a Schrödinger equation

We first recall that the coefficient $a = a_\ell^m$ of the radial part of $\boldsymbol{\xi}$ solves the ODE (4.42)

$$\hat{q}_\ell(r)\, \partial_r^2 a \;+\; q_\ell(r)\, \partial_r a \;+\; \tilde{q}_\ell(r)\, a \;=\; \mathfrak{f}_\ell^m(r)\,. \tag{4.61}$$

We first normalize the first coefficient, so that $a(r) = a_\ell^m(r)$ solves

$$\boxed{-\partial_r^2 a \;+\; \mathfrak{h}_\ell(r)\, \partial_r a \;+\; \mathfrak{g}_\ell(r)\, a \;=\; -\frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)}} \tag{4.62}$$

with function $\mathfrak{h}_\ell(r)$ and $\mathfrak{g}_\ell(r)$ defined as

$$\boxed{\mathfrak{h}_\ell(r) \;:=\; -\frac{q_\ell(r)}{\hat{q}_\ell(r)} \quad , \quad \mathfrak{g}_\ell(r) \;:=\; -\frac{\tilde{q}_\ell(r)}{\hat{q}_\ell(r)}\,.} \tag{4.63}$$

We need to remove the first-order term,

Proposition 6. *The ODE*

$$\hat{q}_\ell(r)\, \partial_r^2 a \;+\; q_\ell(r)\, \partial_r a \;+\; \tilde{q}_\ell(r)\, a \;=\; \mathfrak{f}_\ell^m(r)\,. \tag{4.64}$$

is equivalent to the conjugate ODE

$$-\partial_r^2 \tilde{a} \;+\; V_\ell(r)\, \tilde{a} \;=\; -e^{-\frac{1}{2}\int^r \mathfrak{h}}\, \frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)} \tag{4.65}$$

with unknown

$$\tilde{a}(r) \;:=\; e^{-\frac{1}{2}\int \mathfrak{h}_\ell}\, a(r)\,. \tag{4.66}$$

In the above expression, the new potential V is given as,

$$V_\ell(r) \;=\; \frac{1}{4}\mathfrak{h}_\ell^2(r) \;-\; \frac{1}{2}\partial_r \mathfrak{h}_\ell(r) \;+\; \mathfrak{g}_\ell(r)\,. \tag{4.67}$$

Proof. For lightness of notation, in the current exposition, we drop the subscript ℓ from the coefficients. We have

$$\partial_r\, e^{\frac{1}{2}\int \mathfrak{h}(s)\, ds} \;=\; \frac{\mathfrak{h}(r)}{2}\, e^{\frac{1}{2}\int \mathfrak{h}(s)\, ds} \qquad \Rightarrow \qquad \partial_r^2\, e^{\frac{1}{2}\int \mathfrak{h}(s)\, ds} \;=\; \frac{\mathfrak{h}^2}{4}\, e^{\int \mathfrak{h}} \;+\; \frac{\partial_r \mathfrak{h}}{2} e^{\int \mathfrak{h}} \partial_r \mathfrak{h}. \tag{4.68}$$

Thus

$$\partial_r a \,=\, \partial_r \left(\tilde{a}\, e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\right) \,=\, e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\left(\partial_r \tilde{a} \,+\, \frac{\mathfrak{h}(r)}{2}\,\tilde{a}\right)\,, \tag{4.69}$$

and

$$\begin{aligned}-\partial_r^2 a \,&=\, -e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\,\partial_r^2\tilde{a} \,-\, 2\left(\partial_r e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\right)\partial_r\tilde{a} \,-\, \tilde{a}\,\partial_r^2 e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\\ &=\, e^{\frac{1}{2}\int \mathfrak{h}(s)\,ds}\Big(-\partial_r^2\tilde{a} \,-\, \mathfrak{h}(r)\,\partial_r\tilde{a} \,-\, \frac{\mathfrak{h}^2}{4}\,\tilde{a} \,-\, \frac{\partial_r \mathfrak{h}}{2}\,\tilde{a}\Big)\,.\end{aligned}$$

Substitute a by $a = \tilde{a}e^{\frac{1}{2}\int \mathfrak{h}}$ and the above identities into the ODE (4.62),

$$-\partial_r^2 a \,+\, \mathfrak{h}\,\partial_r a \,+\, \mathfrak{g}\,a \,=\, 0\,,$$

we obtain the conjugate ODE satisfied by $\tilde{a}$

$$e^{\frac{1}{2}\int \mathfrak{h}}\left(-\partial_r^2\tilde{a} \,-\, \mathfrak{h}(r)\,\partial_r\tilde{a} - \left[\frac{1}{2}\partial_r\mathfrak{h} \,+\, \frac{1}{4}\mathfrak{h}^2\right]\tilde{a} \,+\, \mathfrak{h}\,\partial_r\tilde{a} \,+\, \frac{1}{2}\mathfrak{h}^2\tilde{a} \,+\, \mathfrak{g}\,\tilde{a}\right) \,=\, -\frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)}\,.$$

After dividing both sides by $e^{\frac{1}{2}\int \mathfrak{h}}$, we obtain,

$$-\partial_r^2\tilde{a} \,+\, \left(\frac{1}{4}\mathfrak{h}^2 \,-\, \frac{1}{2}\partial_r\mathfrak{h} \,+\, \mathfrak{g}\right)\tilde{a} \,=\, -e^{-\frac{1}{2}\int^r \mathfrak{h}}\,\frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)}\,.$$

□

Explicit expressions for coefficients in the interior For brevity of notation, we use $(\sigma^2)'$ rather than $\mathrm{i}\omega(2\Gamma)'$. For the derivative of $\mathfrak{h}$, we use

$$(k_0^2)' \,=\, \left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)' \,=\, \frac{(\sigma^2)'}{\mathrm{c}_0^2} \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,=\, k_0^2\left(\frac{(\sigma^2)'}{\sigma^2} \,+\, 2\alpha_{\mathrm{c}_0}\right)\,, \tag{4.70}$$

and

$$\left(\frac{1}{k_0^2}\right)' \,=\, -\frac{1}{k_0^4}(k_0^2)' \,=\, -\frac{1}{k_0^2}\left(\frac{(\sigma^2)'}{\sigma^2} \,+\, 2\alpha_{\mathrm{c}_0}\right)\,. \tag{4.71}$$

Proposition 7. *For $r \le r_a$, we have*

$$\mathfrak{h} \,=\, \alpha_{\gamma\mathrm{p}_0} \,-\, \frac{2}{r} \,+\, \frac{\ell(\ell+1)}{k_0^2}\,\frac{2\,\alpha_{\mathrm{c}_0}\,r \,+\, \frac{(\sigma^2)'}{\sigma^2}r \,+\, 2}{r^3 \,-\, \frac{\ell(\ell+1)}{k_0^2}\,r}\,, \tag{4.72}$$

$$\begin{aligned}\mathfrak{h}' \,=\, \alpha_{\gamma\mathrm{p}_0}' \,-\, \frac{2}{r^2} \,+\, \frac{\ell(\ell+1)}{k_0^2}\,&\frac{2\,\alpha_{\mathrm{c}_0} \,+\, 2\alpha_{\mathrm{c}_0}'r \,+\, \frac{(\sigma^2)'}{\sigma^2} \,+\, \left(\frac{(\sigma^2)''}{\sigma^2} \,-\, \left(\frac{(\sigma^2)'}{\sigma^2}\right)^2\right) r}{r^3 \,-\, \frac{\ell(\ell+1)}{k_0^2}\,r}\\ -\, \frac{\ell(\ell+1)}{k_0^2}\,&\frac{2\,\alpha_{\mathrm{c}_0}\,r \,+\, \frac{(\sigma^2)'}{\sigma^2}r \,+\, 2}{\left(r^3 \,-\, \frac{\ell(\ell+1)}{k_0^2}\,r\right)^2}\left(3r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2} \,+\, \frac{\ell(\ell+1)}{k_0^2}\,r\left(2\alpha_{\mathrm{c}_0} \,+\, \frac{(\sigma^2)'}{\sigma^2}\right)\right)\\ -\, \frac{\ell(\ell+1)}{k_0^2}&\left(\frac{(\sigma^2)'}{\sigma^2} \,+\, 2\alpha_{\mathrm{c}_0}\right)\frac{2\,\alpha_{\mathrm{c}_0}\,r \,+\, \frac{(\sigma^2)'}{\sigma^2}r \,+\, 2}{r^3 \,-\, \frac{\ell(\ell+1)}{k_0^2}\,r}\,,\end{aligned} \tag{4.73}$$

and

$$\mathfrak{g} = -k_0^2 + \frac{\Phi_0''}{c_0^2} - \frac{2}{r}\left(\frac{\Phi_0'}{c_0^2} - \alpha_{\rho_0} - 2\alpha_{c_0}\right) + \frac{2+\ell(\ell+1)}{r^2}$$
$$- \frac{\ell(\ell+1)}{k_0^2}\frac{\Phi_0'}{c_0^2}\left(\alpha_{\rho_0} - \frac{\Phi_0'}{c_0^2}\right)\frac{1}{r^2} + \frac{\ell(\ell+1)}{k_0^2}\left(\frac{2}{r^2} - \frac{\Phi_0'}{c_0^2}\frac{1}{r}\right)\frac{2\,\alpha_{c_0}\,r + 2 + \frac{(\sigma^2)'}{\sigma^2}r}{r^2 - \frac{\ell(\ell+1)}{k_0^2}}. \tag{4.74}$$

Explicit expressions for coefficients in the atmosphere The derivation is given in Appendix B.

Proposition 8. *For $r \geq r_a$, we have*

$$\mathfrak{h} = \alpha_{\rho_0} - \frac{2}{r} + \ell(\ell+1)\frac{(k_0^2)'\,r^2 + 2\,k_0^2\,r - E_{he} - r\frac{\Phi_0''}{c_0^2}}{\left(k_0^2 r^2 - \ell(\ell+1) - rE_{he}\right)\,(k_0^2\,r^2 - r\,E_{he})}, \tag{4.75}$$

$$\mathfrak{g} = -k_0^2 + \frac{2\,(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma})}{r} + \frac{2}{r^2} + \ell(\ell+1)\frac{k_0^2 + \frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} - \alpha_{\rho_0}\right) - \frac{\Phi_0''}{c_0^2}}{k_0^2\,r^2 - r\,E_{he}} \tag{4.76a}$$

$$+ \frac{\Phi_0''}{c_0^2} + \ell(\ell+1)\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 + 2\,k_0^2\,r - E_{he} - r\frac{\Phi_0''}{c_0^2}}{\left(k_0^2 r^2 - \ell(\ell+1) - rE_{he}\right)\,(k_0^2\,r^2 - r\,E_{he})}, \tag{4.76b}$$

and, under the hypothesis of constant attenuation,

$$\mathfrak{h}' = \frac{2}{r^2} + \ell(\ell+1)\frac{2\,k_0^2 - \frac{4\pi G}{c_0^2}r\,\rho_0' - 2\frac{\Phi_0'}{c_0^2}\frac{1}{r}}{\left(k_0^2 r^2 - \ell(\ell+1) - rE_{he}\right)\,(k_0^2\,r^2 - r\,E_{he})},$$
$$- \frac{\ell(\ell+1)\left(2\,k_0^2\,r - E_{he} - r\frac{\Phi_0''}{c_0^2}\right)\left(2\,k_0^2\,r - E_{he} - r\frac{\Phi_0''}{c_0^2}\right)\left(2k_0^2 r^2 - \ell(\ell+1) - 2rE_{he}\right)}{\left(k_0^2 r^2 - \ell(\ell+1) - rE_{he}\right)^2\,(k_0^2\,r^2 - r\,E_{he})^2}. \tag{4.77}$$

5 Alternative to obtain the system of equations in the solar interior

Another way to solve the Galbrun equation in the interior of the Sun is to work directly with the form of the equation (3.31) given in Proposition 2 under the assumption of hydrostatic equilibrium (3.14).

$$-\rho_0\,\sigma^2\,\boldsymbol{\xi} + \nabla\delta_{\mathrm{p}} + \delta_\rho\,\nabla\Phi_0 = \mathbf{f}\,; \tag{5.1a}$$
$$\delta_\rho = -(\nabla\rho_0)\cdot\boldsymbol{\xi} - \rho_0\nabla\cdot\boldsymbol{\xi}\,; \tag{5.1b}$$
$$\delta_{\mathrm{p}} = -\boldsymbol{\xi}\cdot\nabla\mathrm{p}_0 - \rho_0\,c_0^2\,\nabla\cdot\boldsymbol{\xi}\,. \tag{5.1c}$$

Here we ignore the perturbation in gravitation potential δ_Φ^{E}. For the rest of this section, to alleviate the notation, we will drop the superscript E from the Eulerian perturbation $\delta_\bullet^{\mathsf{E}}$. The main ideas are from [11]. The main difference is that instead of working with vector harmonic bases ($\mathbf{P}_\ell^m$, $\mathbf{C}_\ell^m$, and $\mathbf{B}_\ell^m$ cf. (2.20)), the equation is rewritten in terms of only scalar unknowns and thus have the usual harmonic expansion in Y_ℓ^m.

We first make some initial remarks on the special feature of (5.1a). Decompose the unknown $\boldsymbol{\xi}$ and the external source into the radial component and tangential one,

$$\boldsymbol{\xi} = \xi_r\,\mathbf{e}_r + \boldsymbol{\xi}_h \quad , \quad \xi_r := \boldsymbol{\xi}\cdot\mathbf{e}_r\,;$$
$$\mathbf{f} = f_r\mathbf{e}_r + \mathbf{f}_h \quad , \quad f_r := \mathbf{f}\cdot\mathbf{e}_r\,. \tag{5.2}$$

We also decompose ∇ into radial and tangential component, in particular for a scalar function f,

$$\nabla f \;=\; (\partial_r f)\,\mathbf{e}_r \;+\; \frac{1}{r}\nabla_{\mathbb{S}^2} f\,.$$

The decomposition of (5.1a) along $\mathbf{e}_r$ and tangential one gives, after dividing both sides by ρ_0,

$$-\,\sigma^2\,\xi_r \;+\; \frac{1}{\rho_0}\partial_r\,\delta_{\mathrm{p}} \;+\; \frac{\delta_\rho}{\rho_0}\,\Phi_0' \;=\; \frac{f_r}{\rho_0}\,; \tag{5.3a}$$

$$-\,\sigma^2\,\boldsymbol{\xi}_h \;+\; \frac{1}{r\,\rho_0}\nabla_{\mathbb{S}^2}\delta_{\mathrm{p}} \;+\; \frac{\delta_\rho}{\rho_0\,r}\nabla_{\mathbb{S}^2}\Phi_0 \;=\; \frac{\mathbf{f}_h}{\rho_0}\,. \tag{5.3b}$$

Under spherical symmetry of the background, we solve for scalar unknowns which can be represented in harmonic expansion; in particular,

$$\begin{aligned}\xi_r(\mathbf{x}) \;&=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(r)\;\mathrm{Y}_\ell^m(\theta,\phi)\,;\\ \delta_\rho(\mathbf{x}) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell} d^\ell(r)\;\mathrm{Y}_\ell^m(\theta,\phi) \quad &, \quad \delta_{\mathrm{p}}(\mathbf{x}) \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell} e^\ell(r)\;\mathrm{Y}_\ell^m(\theta,\phi)\,.\end{aligned} \tag{5.4}$$

With $\nabla_{\mathbb{S}^2}\Phi_0 = 0$ under spherical symmetry assumption, and if $\mathbf{f}_h = 0$, then (5.3b) implies that the tangential part $\boldsymbol{\xi}_h$ exists solely along $\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m$, i.e. for some $t_\ell^m(r)$,

$$\boldsymbol{\xi}_h \;=\; \frac{1}{\sigma^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell} t_\ell^m(r)\;\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m(\theta,\phi)\,. \tag{5.5}$$

Assumptions In addition to symmetric background ρ_0, γ and c_0, we assume that the radial and tangential divergence part of the source can be represented in harmonic expansion, i.e.

$$\begin{aligned}f_r \;=\; \mathbf{f}\cdot\mathbf{e}_r \quad , \quad \mathbf{f}_h \;=\; \mathbf{f} \;-\; f_r\mathbf{e}_r \quad , \quad \tilde{f}_h \;:=\; \nabla_{\mathbb{S}^2}\cdot\mathbf{f}_h\,;\\ f_r \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m(r)\;\mathrm{Y}_\ell^m(\theta,\phi) \quad , \quad \tilde{f}_h \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[f_h]_\ell^m(r)\;\mathrm{Y}_\ell^m(\theta,\phi)\,.\end{aligned} \tag{5.6}$$

5.1 Approach 1 - A first order system

Given known quantities

$$\rho_0 \quad , \quad \mathrm{c}_0 \quad , \quad \mathrm{p}_0 \quad , \quad \Phi_0 \tag{5.7}$$

Thus we look for unknowns

$$\boldsymbol{\xi} \quad , \quad \delta_{\mathrm{p}} \quad , \quad \delta_\rho \tag{5.8}$$

which solves (5.1), and with the property,

$$\begin{aligned}\xi_r \;=\; \boldsymbol{\xi}\cdot\mathbf{e}_r \quad , \quad \boldsymbol{\xi}_h = \boldsymbol{\xi} - \xi_r\mathbf{e}_r \quad , \quad \tilde{\xi}_h \;:=\; \nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\,;\\ \xi_r \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(r)\,\mathrm{Y}_\ell^m(\theta,\phi) \quad , \quad \tilde{\xi}_h \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\tilde{b}_\ell^m(r)\,\mathrm{Y}_\ell^m(\theta,\phi)\,.\end{aligned} \tag{5.9}$$

Step 1 By taking the difference of the last two equations in (5.1), we obtain an equivalent system, which under spherical symmetry simplifes to

$$-\,\sigma^2\,\xi_r \;+\; \frac{1}{\rho_0}\partial_r\delta_{\mathrm{p}} \;+\; \frac{\delta_\rho}{\rho_0}\,\Phi_0' \;=\; \frac{f_r}{\rho_0}\,; \tag{5.10a}$$

$$-\,\sigma^2\,\boldsymbol{\xi}_h \;+\; \frac{1}{r\,\rho_0}\nabla_{\mathbb{S}^2}\delta_{\mathrm{p}} \;=\; \frac{\mathbf{f}_h}{\rho_0}\,; \tag{5.10b}$$

$$\delta_\rho \;=\; -\rho_0'\,\xi_r \;-\; \rho_0\left(\partial_r\xi_r + \frac{2}{r}\xi_r \;+\; \frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\right) \tag{5.10c}$$

$$\mathrm{c}_0^2\delta_\rho \;-\; \delta_{\mathrm{p}} \;=\; \xi_r\cdot\left(\mathrm{p}_0' \;-\; \mathrm{c}_0^2\rho_0'\right). \tag{5.10d}$$

In the second-to-last equality, we have used $\nabla\cdot\boldsymbol{\xi} = \partial_r\xi_r + \frac{2}{r}\xi_r + \frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h$.

Step 2 : Take $\nabla_{\mathbb{S}^2}\cdot$ of (5.10b)

$$-\sigma^2\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h \;+\; \frac{1}{r\,\rho_0}\Delta_{\mathbb{S}^2}\delta_{\mathrm{p}} \;=\; \frac{\nabla_{\mathbb{S}^2}\cdot\mathbf{f}_h}{\mathrm{p}_0}\,. \tag{5.11}$$

We can use (5.10c) to eliminate $\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h$ from (5.11) and obtain

$$\sigma^2\left(\frac{1}{\rho_0}\delta_\rho \;+\; \frac{\rho_0'}{\rho_0}\,\xi_r \;+\; \partial_r\xi_r + \frac{2}{r}\xi_r\right) \;+\; \frac{1}{r^2\,\rho_0}\Delta_{\mathbb{S}^2}\delta_{\mathrm{p}} \;=\; \frac{\nabla_{\mathbb{S}^2}\cdot\mathbf{f}_h}{r\,\mathrm{p}_0}\,. \tag{5.12}$$

Step 3 : We will use (5.12) together with (5.10a) and (5.10d) to solve for the three scalar unknowns ξ_r, δ_ρ and δ_{p}. Obtain equations in terms of the coefficients of the harmonic expansions of these unknowns, listed in order of (5.10a), (5.12) and (5.10d),

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(-\sigma^2\,a_\ell^m(r) \;+\; \frac{1}{\rho_0}\partial_r e_\ell^m \;+\; \frac{\Phi_0'}{\rho_0}\,d_\ell^m\right)\mathrm{Y}_\ell^m \;=\; \frac{1}{\rho_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m\,\mathrm{Y}_\ell^m\,; \tag{5.13a}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\sigma^2\left(\frac{1}{\rho_0}d_\ell^m \;+\; \frac{\rho_0'}{\rho_0}\,a_\ell^m \;+\; \partial_r a_\ell^m + \frac{2}{r}a_\ell^m\right) \;-\; \frac{\ell(\ell+1)}{r^2\,\rho_0}e_\ell^m\right)\mathrm{Y}_\ell^m \;=\; \frac{1}{r\,\rho_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}[f_h]_\ell^m\,\mathrm{Y}_\ell^m\,; \tag{5.13b}$$

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(-\mathrm{c}_0^2\,d_\ell^m \;+\; e_\ell^m \;+\; \left(\mathrm{p}_0' \,-\, \mathrm{c}_0^2\,\rho_0'\right)a_\ell^m\right)\mathrm{Y}_\ell^m \;=\; 0. \tag{5.13c}$$

In the second equation, we have used $\Delta_{\mathbb{S}^2}\mathrm{Y}_\ell^m = -\ell(\ell+1)\mathrm{Y}_\ell^m$. We thus obtain a first order equation

$$\begin{pmatrix} \frac{1}{\rho_0} & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & 0\end{pmatrix}\partial_r\begin{pmatrix} e_\ell^m \\ a_\ell^m \\ d_\ell^m\end{pmatrix} + \begin{pmatrix} 0 & -\sigma^2 & \frac{\Phi_0'}{\rho_0} \\ -\frac{\ell(\ell+1)}{r^2\,\mathrm{p}_0} & \sigma^2\left(\frac{\rho_0'}{\rho_0} + \frac{2}{r}\right) & \frac{\sigma^2}{\rho_0} \\ \frac{1}{\mathrm{c}_0^2} & \frac{\mathrm{p}_0'}{\mathrm{c}_0^2} - \rho_0' & -1 \end{pmatrix}\begin{pmatrix} e_\ell^m \\ a_\ell^m \\ d_\ell^m\end{pmatrix} = \frac{1}{r\,\rho_0}\begin{pmatrix} r\,f_\ell^m \\ [f_h]_\ell^m \\ 0\end{pmatrix} \tag{5.14}$$

We can further eliminate d_ℓ^m

$$\mathbb{A}\,\partial_r\begin{pmatrix} e_\ell^m \\ a_\ell^m\end{pmatrix} \;+\; \mathbb{B}\begin{pmatrix} e_\ell^m \\ a_\ell^m\end{pmatrix} \;=\; \frac{1}{r\,\rho_0}\begin{pmatrix} r\,f_\ell^m \\ [f_h]_\ell^m\end{pmatrix} \tag{5.15}$$

with

$$\mathbb{A} \;=\; \begin{pmatrix} \frac{1}{\rho_0} & 0 \\ 0 & \sigma^2\end{pmatrix}\quad,\quad \mathbb{B} \;=\; \begin{pmatrix} \frac{1}{\mathrm{c}_0^2}\frac{\Phi_0'}{\rho_0} & -\sigma^2 + \frac{\Phi_0'}{\rho_0}\left(\frac{\mathrm{p}_0'}{\mathrm{c}_0^2} - \rho_0'\right) \\ \frac{\sigma^2}{\rho_0\mathrm{c}_0^2} - \frac{\ell(\ell+1)}{r^2\,\mathrm{p}_0} & \sigma^2\left(\frac{\mathrm{p}_0'}{\rho_0\mathrm{c}_0^2} + \frac{2}{r}\right)\end{pmatrix}. \tag{5.16}$$

Remark 12. *Defining the buoyancy frequency N as*

$$N^2 = \Phi_0'\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0} - \frac{\rho_0'}{\rho_0}\right), \tag{5.17}$$

and the characteristic acoustic frequency S_l (also called Lamb frequency by [35]) by

$$S_l^2 = \frac{\ell(\ell+1)\mathrm{c}_0^2}{r^2}, \tag{5.18}$$

we can rewrite the matrix $\mathbb{B}$ *as*

$$\mathbb{B} \;=\; \begin{pmatrix} -\dfrac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0} & N^2 \;-\sigma^2 \\[2ex] \dfrac{1}{\rho_0\mathrm{c}_0^2}\left(\sigma^2 - S_l^2\right) & \sigma^2\left(\dfrac{\mathrm{p}_0'}{\gamma\mathrm{p}_0} \;+\; \dfrac{2}{r}\right) \end{pmatrix}, \tag{5.19}$$

where we have used the hydrostatic equilibrium $\rho_0\Phi_0' + \mathrm{p}_0' = 0$ *and adiabaticity* $\rho_0\mathrm{c}_0^2 = \gamma\mathrm{p}_0$*. This system is equivalent to Eqs.(4.61) and (4.62) in [11] and Eqs. (14.2) (14.3) in [35].* △

5.1.1 Recovering the decoupled ODE for the radial displacement

As a sanity check, we rederive the scalar wave equation satisfied by the radial displacement from the system (5.15). For simplicity we consider the homogeneous equations. From the second line of the system one gets

$$e_l^m = \frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\left(\partial_r a_l^m + \left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0} + \frac{2}{r}\right)a_l^m\right), \tag{5.20}$$

while the first line can be written as

$$\frac{1}{\rho_0}\partial_r e_l^m - \frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0}e_l^m + (N^2-\sigma^2)a_l^m = 0. \tag{5.21}$$

Combining these two equations gives

$$\begin{aligned} &\frac{1}{\rho_0}\partial_r\left[\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\left(\partial_r a_l^m + \left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right)a_l^m\right)\right] \\ &-\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0}\left[\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\left(\partial_r a_l^m + \left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right)a_l^m\right)\right] + (N^2-\sigma^2)a_l^m = 0, \end{aligned} \tag{5.22}$$

which is the second order equation satisfied by the coefficients a_l^m that we want to write on the form

$$A\partial_r^2 a_l^m + B\partial_r a_l^m + C a_l^m = 0. \tag{5.23}$$

The coefficient A is given by

$$A = \frac{\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2} = \frac{\sigma^2}{C_{22}}. \tag{5.24}$$

To evaluate B, we need to compute

$$\frac{1}{\rho_0}\partial_r\left(\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\right) = \frac{1}{\rho_0}\frac{\partial_r\left(\rho_0\mathrm{c}_0^2\sigma^2\right)}{S_l^2-\sigma^2} - \frac{\mathrm{c}_0^2\sigma^2\partial_r(S_l^2-\sigma^2)}{(S_l^2-\sigma^2)^2} \tag{5.25}$$

$$= \frac{-\alpha\mathrm{c}_0^2\sigma^2 + 2\mathrm{c}_0\mathrm{c}_0'\sigma^2 + 2\mathrm{i}\omega\mathrm{c}_0^2\Gamma'}{S_l^2-\sigma^2} + \frac{\mathrm{c}_0^2\sigma^2\left(2\mathrm{i}\omega\Gamma' + 2S_l^2\left(\alpha_{\mathrm{c}_0}+\frac{1}{r}\right)\right)}{(S_l^2-\sigma^2)^2}, \tag{5.26}$$

where we used that

$$\partial_r\left(S_l^2\right) = \ell(\ell+1)\left(\frac{2\mathrm{c}_0\mathrm{c}_0'}{r^2} - \frac{2\mathrm{c}_0^2}{r^3}\right) = -2S_l^2\left(\alpha_{\mathrm{c}_0} + \frac{1}{r}\right). \tag{5.27}$$

Thus,

$$rC_{22}(r)B = rC_{22}(r)\left(\frac{1}{\rho_0}\partial_r\left(\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\right) + \frac{2}{r}\frac{\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\right) \tag{5.28}$$

$$= r\left(-\alpha\sigma^2 - 2\alpha_{\mathrm{c}_0}\sigma^2 + 2\mathrm{i}\omega\Gamma'\right) + \frac{\sigma^2\left(2\mathrm{i}\omega\Gamma' r + 2S_l^2\left(\alpha_{\mathrm{c}_0}r+1\right)\right)}{(S_l^2-\sigma^2)} + 2\sigma^2, \tag{5.29}$$

$$= -\alpha_{\gamma\mathrm{p}_0}\sigma^2 r + 2\mathrm{i}\omega\Gamma' r\frac{S_l^2}{S_l^2-\sigma^2} + \frac{2\sigma^2 S_l^2\left(\alpha_{\mathrm{c}_0}r+1\right)}{(S_l^2-\sigma^2)} + 2\sigma^2, \tag{5.30}$$

where we used the relation between the different scale heights (6.3). Replacing S_l^2 by its definition (5.18), it follows that

$$rC_{22}(r)B = -\alpha_{\gamma \mathrm{p}_0}\sigma^2 r + 2\sigma^2 - \ell(\ell+1)\mathrm{c}_0^2 \frac{r\left(2\sigma^2\alpha_{\mathrm{c}_0} + 2\mathrm{i}\omega\Gamma'\right) + 2\sigma^2}{\sigma^2 r^2 - \ell(\ell+1)\mathrm{c}_0^2}, \tag{5.31}$$

$$= rC_{22}(r)q(r)\mathrm{c}_0^2, \tag{5.32}$$

where the expression for $rC_{22}q$ is given by (4.53).

The coefficient C is given by

$$C = \frac{1}{\rho_0}\partial_r\left[\left(\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\right)\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right)\right] - \frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0}\left[\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right)\right] + (N^2-\sigma^2) \tag{5.33}$$

$$= \frac{1}{\rho_0}\partial_r\left[\frac{\rho_0\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\right]\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right) + \frac{\mathrm{c}_0^2\sigma^2}{S_l^2-\sigma^2}\left(\frac{\mathrm{p}_0''}{\gamma\mathrm{p}_0}+\frac{\alpha_{\gamma\mathrm{p}_0}\mathrm{p}_0'}{\gamma\mathrm{p}_0}-\frac{2}{r^2}\right)$$
$$- \frac{\mathrm{p}_0'\mathrm{c}_0^2\sigma^2}{\gamma\mathrm{p}_0(S_l^2-\sigma^2)}\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right) - \sigma^2 + \Phi_0'\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}-\frac{\rho_0'}{\rho_0}\right). \tag{5.34}$$

Using the previous computation for $rC_{22}B$, we obtain

$$r^2C_{22}(r)C = \left(-\alpha_{\gamma\mathrm{p}_0}\sigma^2 r + \frac{\alpha_{\mathrm{p}_0}}{\gamma}\sigma^2 r + 2\mathrm{i}\omega\Gamma' r\frac{S_l^2}{S_l^2-\sigma^2} + \frac{2\sigma^2 S_l^2\left(\alpha_{\mathrm{c}_0}r+1\right)}{S_l^2-\sigma^2}\right)\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}r+2\right) \tag{5.35}$$

$$+\sigma^2\left(\frac{\mathrm{p}_0''}{\gamma\mathrm{p}_0}r^2+\frac{\alpha_{\gamma\mathrm{p}_0}\mathrm{p}_0'}{\gamma\mathrm{p}_0}r^2-2\right) - \sigma^2 r^2\frac{S_l^2-\sigma^2}{\mathrm{c}_0^2} + \Phi_0'\left(-\frac{\alpha_{\mathrm{p}_0}}{\gamma}+\alpha_{\rho_0}\right)r^2\frac{S_l^2-\sigma^2}{\mathrm{c}_0^2}. \tag{5.36}$$

It follows that

$$r^2C_{22}(r)C = \frac{\sigma^2 r^2}{\mathrm{c}_0^2}\left(-\sigma^2+\Phi_0''\right) + 2\sigma^2\left(-r\alpha_{\gamma\mathrm{p}_0}+\frac{\alpha_{\mathrm{p}_0}}{\gamma}r-1\right)$$
$$+\left(2-\frac{\alpha_{\mathrm{p}_0}r}{\gamma}\right)S_l^2\frac{2\sigma^2(1+\alpha_{\mathrm{c}_0}r)+2\mathrm{i}\omega\Gamma' r}{S_l^2-\sigma^2} + \ell(\ell+1)\left(-\sigma^2+\Phi_0'\left(-\frac{\alpha_{\mathrm{p}_0}}{\gamma}+\alpha_{\rho_0}\right)\right), \tag{5.37}$$

where we gathered and simplified the terms of order r^2 using the definition of Φ_0' and Φ_0'' from (6.9) and (6.11) and used the definition of S_l ((5.18)) for the last term. Replacing S_l and using (6.9) to introduce Φ_0'

$$r^2C_{22}(r)C = \frac{\sigma^2 r^2}{\mathrm{c}_0^2}\left(-\sigma^2+\Phi_0''\right) + 2\sigma^2\left(-r\alpha_{\gamma\mathrm{p}_0}+\frac{\alpha_{\mathrm{p}_0}}{\gamma}r-1\right)$$
$$-\ell(\ell+1)\left(2-\frac{\alpha_{\mathrm{p}_0}r}{\gamma}\right)\frac{2r\left(\sigma^2\alpha_{\mathrm{c}_0}+\mathrm{i}\omega\Gamma'\right)+2\sigma^2}{\sigma^2r^2-\ell(\ell+1)} + \ell(\ell+1)\left(-\sigma^2+\Phi_0'\left(-\frac{\Phi_0'}{\mathrm{c}_0^2}+\alpha_{\rho_0}\right)\right), \tag{5.38}$$

$$= r^2C_{22}(r)\mathrm{c}_0^2\tilde{q}(r). \tag{5.39}$$

The expression of $r^2C_{22}\tilde{q}$ is given by (4.55). The obtained decoupled system is thus the same than the one derived in Subsection 4.4.1.

5.1.2 Decoupled system satisfied by the pressure

We can proceed in a similar manner than in the previous section, in order to eliminate the radial displacement and obtain a decoupled ODE for the Lagrangian perturbation of the pressure δ_{p}. We first express the coefficient a_l^m as a function of e_l^m using the second line of the system (5.15)

$$a_l^m = \frac{1}{\rho_0(\sigma^2-N^2)}\partial_r e_l^m - \frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0(\sigma^2-N^2)}e_l^m. \tag{5.40}$$

Here, we suppose for simplicity that $\Gamma \neq 0$ so that the term $(\sigma^2 - N^2)$ does not vanish. Then the second line becomes

$$\frac{\sigma^2}{\rho_0(\sigma^2-N^2)}\partial_r^2 e_l^m + \sigma^2\partial_r\left(\frac{1}{\rho_0(\sigma^2-N^2)}\right)\partial_r e_l^m - \sigma^2\partial_r\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0(\sigma^2-N^2)}\right)e_l^m \tag{5.41}$$

$$-\frac{\sigma^2\mathrm{p}_0'}{\gamma\mathrm{p}_0\rho_0(\sigma^2-N^2)}e_l^m + \frac{1}{\rho_0\mathrm{c}_0^2}(\sigma^2-S_l^2)e_l^m + \frac{\sigma^2}{\rho_0(\sigma^2-N^2)}\left(\frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}+\frac{2}{r}\right)\left(\partial_r e_l^m - \frac{\mathrm{p}_0'}{\gamma\mathrm{p}_0}e_l^m\right) = 0. \tag{5.42}$$

Multiplying by $\rho_0(\sigma^2 - N^2)$ leads to

$$\partial_r^2 e_l^m + \left(\alpha_{\rho_0} - \frac{(\sigma^2 - N^2)'}{\sigma^2 - N^2} - \frac{\alpha_{\mathrm{p}_0}}{\gamma} + \frac{2}{r}\right) \partial_r e_l^m \tag{5.43}$$

$$+ \left(\partial_r \left(\frac{\mathrm{p}_0'}{\gamma \mathrm{p}_0}\right) + \frac{\mathrm{p}_0'}{\gamma \mathrm{p}_0}\left(\alpha_{\rho_0} - \frac{(\sigma^2 - N^2)'}{\sigma^2 - N^2} - 1 - \frac{\mathrm{p}_0'}{\gamma \mathrm{p}_0} - \frac{2}{r}\right) + \frac{\sigma^2 - N^2}{\sigma^2 \mathrm{c}_0^2}\left(\sigma^2 - S_l^2\right)\right) e_l^m = 0. \tag{5.44}$$

This is a second order ODE satisfied by the Lagrangian perturbation of the pressure. This corresponds to the scalar problem proposed by [20] for which the boundary conditions were proposed in [3, 17] and which was studied theoretically in [5, 6]. The main difference is the incorporation of the gravity term. One can recover the equation from [20] by setting $N^2 = 0$ and $\mathrm{p}_0' = 0$ in (5.44).

From the knowledge of e_l^m, one can recover the radial part of the displacement using (5.40) and the horizontal part from (4.45). In the framework of [20] (without gravity), these coefficients are given by

$$a_l^m = \frac{1}{\rho_0 \sigma^2} \partial_r e_l^m, \tag{5.45}$$

$$b_l^m = \frac{\sqrt{\ell(\ell+1)}}{k_h^2 - k_0^2} \frac{\partial_r \left(r^2 a_l^m\right)}{r^3}, \tag{5.46}$$

where $k_h = \sqrt{\ell(\ell+1)}/r$ is the horizontal wavenumber and $k_0^2 = \sigma^2/\mathrm{c}_0^2$ is the local wavenumber. Note that in the presence of attenuation the denominator never vanishes and the coefficient b_l^m can be computed.

5.2 Approach 2 - a second order system

From the form of δ_ρ in (5.10c), which is similar for δ_{p}, the radial equations of motion are in terms of unknowns of ξ_r and $\nabla_{\mathbb{S}^2} \cdot \boldsymbol{\xi}_h$. Recall the definition of the radial and the tangential divergence of the tangential part,

$$\begin{aligned} f_r &= \mathbf{f} \cdot \mathbf{e}_r \quad , \quad \mathbf{f}_h = \mathbf{f} - f_r \, \mathbf{e}_r \quad , \quad \tilde{\xi}_h := \nabla_{\mathbb{S}^2} \cdot \mathbf{f}_h \, ; \\ \xi_r &= \boldsymbol{\xi} \cdot \mathbf{e}_r \quad , \quad \boldsymbol{\xi}_h = \boldsymbol{\xi} - \xi_r \, \mathbf{e}_r \quad , \quad \tilde{\xi}_h := \nabla_{\mathbb{S}^2} \cdot \boldsymbol{\xi}_h \, . \end{aligned}$$

Taking $\nabla_{\mathbb{S}^2} \cdot$ of the tangential equation (5.3b) provides another equation in these two variables.

$$-\rho_0 \, \sigma^2 \, \nabla_{\mathbb{S}^2} \cdot \boldsymbol{\xi}_h \; + \; \frac{1}{r} \Delta_{\mathbb{S}^2} \delta_{\mathrm{p}} = \nabla_{\mathbb{S}^2} \cdot \mathbf{f}_h \, ,$$

which, in terms of $\tilde{\xi}_h$ and $\tilde{f}_h$, is

$$-\rho_0 \, \sigma^2 \, \tilde{\xi}_h \; + \; \frac{1}{r} \, \Delta_{\mathbb{S}^2} \delta_{\mathrm{p}} \; = \; \tilde{f}_h \, . \tag{5.47}$$

In this approach, one solely works with two scalar unknowns ξ_r and $\tilde{\xi}_h$. The similarity as in Approach 1 is that they are both scalar unknowns, and are supposed to have an expansion in spherical harmonics. At the end, we will identify the resulting system with (4.33).

To solve for ξ_r and $\tilde{\xi}_h$, we use the above equation together with the radial part of the equation of motion,

$$-\,\sigma^2 \, \xi_r \; + \; \frac{1}{\rho_0} \partial_r \delta_{\mathrm{p}} \; + \; \frac{\delta_\rho}{\rho_0} \, \Phi_0' \; = \; \frac{f_r}{\rho_0} \, , \tag{5.48a}$$

$$-\,\sigma^2 \, \tilde{\xi}_h \; + \; \frac{1}{r \, \rho_0} \Delta_{\mathbb{S}^2} \delta_{\mathrm{p}} \; = \; \frac{\tilde{f}_h}{\rho_0} \, , \tag{5.48b}$$

with

$$\delta_\rho \; = \; -\left(\mathrm{p}_0' \; + \; \frac{2\rho_0}{r}\right) \xi_r \; - \; \rho_0 \, \partial_r \xi_r \; - \; \frac{\rho_0}{r} \tilde{\xi}_h \, ; \tag{5.49a}$$

$$\delta_{\mathrm{p}} \; = \; -\left(\mathrm{p}_0' \; + \; \frac{2\,\rho_0\,\mathrm{c}_0^2}{r}\right) \xi_r \; - \; \rho_0 \, \mathrm{c}_0^2 \, \partial_r \xi_r \; - \; \frac{\rho_0 \, \mathrm{c}_0^2}{r} \tilde{\xi}_h \, . \tag{5.49b}$$

We next obtain explicit equation for the coefficients a_ℓ^m and $\tilde{b}_\ell^m$.

We first consider the tangential equation (5.48b). From (5.49b), using $\Delta_{\mathbb{S}^2}\mathrm{Y}_\ell^m = -\ell(\ell+1)\,\mathrm{Y}_\ell^m$, we have

$$\frac{1}{\rho_0}\Delta_{\mathbb{S}^2}\,\delta_{\mathrm{p}} \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\ell(\ell+1)\left(\left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\right)a_\ell^m \;+\; \mathrm{c}_0^2\,\partial_r a_\ell^m \;+\; \frac{\mathrm{c}_0^2}{r}\hat{b}_\ell^m\right)\mathrm{Y}_\ell^m\,. \tag{5.50}$$

Substituting this expression into (5.48b), we obtain

$$\begin{aligned}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\Bigg(\left(-\sigma^2\;+\;\frac{\ell(\ell+1)\,\mathrm{c}_0^2}{r^2}\right)\hat{b}_\ell^m \;+\; \frac{\ell(\ell+1)}{r}\left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\right)a_\ell^m\\ +\;\frac{\ell(\ell+1)}{r}\mathrm{c}_0^2\partial_r a_\ell^m\Bigg)\mathrm{Y}_\ell^m \;=\; \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{[f_h]_\ell^m}{\rho_0}\,\mathrm{Y}_\ell^m\,.\end{aligned} \tag{5.51}$$

Dividing both sides by c_0^2, we have, on each mode (ℓ,m) the equation (which does not depend on m)

$$\left(-\frac{\sigma^2}{\mathrm{c}_0^2}\;+\;\frac{\ell(\ell+1)}{r^2}\right)\hat{b}_\ell^m \;+\; \frac{\ell(\ell+1)}{r}\left(\frac{\mathrm{p}_0'}{\mathrm{c}_0^2\,\rho_0}\;+\;\frac{2}{r}\right)a_\ell^m \;+\; \frac{\ell(\ell+1)}{r}\partial_r a_\ell^m \;=\; \frac{[f_h]_\ell^m}{\mathrm{c}_0^2\,\rho_0}\,; \tag{5.52}$$

We now consider the radial equation (5.48a). From (5.49b), we have

$$\begin{aligned}\frac{1}{\rho_0}\partial_r\delta_{\mathrm{p}} =& -\left(\frac{\mathrm{p}_0''}{\rho_0}\;+\;\frac{1}{\rho_0}\left(\frac{2\rho_0\,\mathrm{c}_0^2}{r}\right)'\right)\xi_r \;-\; \frac{(\rho_0\,\mathrm{c}_0^2)'}{\rho_0}\,\partial_r\xi_r \;-\; \frac{1}{\rho_0}\left(\frac{\rho_0\,\mathrm{c}_0^2}{r}\right)'\tilde{\xi}_h\\ &-\;\left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\right)\partial_r\xi_r \;-\; \mathrm{c}_0^2\,\partial_r^2\xi_r \;-\; \frac{\mathrm{c}_0^2}{r}\partial_r\tilde{\xi}_h\\ =& -\left(\frac{\mathrm{p}_0''}{\rho}\;+\;\frac{1}{\rho_0}\left(\frac{2\rho_0\,\mathrm{c}_0^2}{r}\right)'\right)\xi_r \;-\; \left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\;+\;\frac{(\rho_0\,\mathrm{c}_0^2)'}{\rho_0}\right)\partial_r\xi_r\\ &-\mathrm{c}_0^2\,\partial_r^2\xi_r \;-\; \frac{\mathrm{c}_0^2}{r}\partial_r\tilde{\xi}_h \;-\; \frac{1}{\rho_0}\left(\frac{\rho_0\,\mathrm{c}_0^2}{r}\right)'\tilde{\xi}_h\,.\end{aligned} \tag{5.53}$$

We further expand the derivative of $\left(\frac{\rho_0\mathrm{c}_0^2}{r}\right)'$,

$$\begin{aligned}\frac{1}{\rho_0}\partial_r\delta_{\mathrm{p}} =& -\left(\frac{\mathrm{p}_0''}{\rho}\;+\;\frac{2(\rho_0\,\mathrm{c}_0^2)'}{\rho_0\,r}\;-\;\frac{2\mathrm{c}_0^2}{r^2}\right)\xi_r \;-\; \left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\;+\;\frac{(\rho_0\,\mathrm{c}_0^2)'}{\rho_0}\right)\partial_r\xi_r\\ &-\mathrm{c}_0^2\,\partial_r^2\xi_r \;-\; \frac{\mathrm{c}_0^2}{r}\partial_r\tilde{\xi}_h \;-\; \left(\frac{(\rho_0\,\mathrm{c}_0^2)'}{r\,\rho_0}\;-\;\frac{\mathrm{c}_0^2}{r^2}\right)\tilde{\xi}_h\,.\end{aligned} \tag{5.54}$$

On the other hand, from (5.49a),

$$\frac{\Phi_0'}{\rho_0}\delta_\rho \;=\; -\frac{\Phi_0'}{\rho_0}\left(\mathrm{p}_0'\;+\;\frac{2\rho_0}{r}\right)\xi_r \;-\; \Phi_0'\;\partial_r\xi_r \;-\; \frac{\Phi_0'}{r}\tilde{\xi}_h\,.$$

Substitute the above expressions for $\frac{1}{\rho_0}\partial_r\delta_{\mathrm{p}}$ and $\frac{\Phi_0'}{\rho_0}\delta_\rho$ into (5.48a), we obtain the following equations (as coefficients of Y_ℓ^m on each level),

$$\begin{aligned}-\mathrm{c}_0^2\,\partial_r^2 a_\ell^m \;-\; \left(\sigma^2\;+\;\frac{\mathrm{p}_0''}{\rho}\;+\;\frac{2(\rho_0\,\mathrm{c}_0^2)'}{\rho_0\,r}\;-\;\frac{2\mathrm{c}_0^2}{r^2}\;+\;\frac{\Phi_0'}{\rho_0}\left(\mathrm{p}_0'\;+\;\frac{2\rho_0}{r}\right)\right)a_\ell^m\\ -\left(\frac{\mathrm{p}_0'}{\rho_0}\;+\;\frac{2\,\mathrm{c}_0^2}{r}\;+\;\frac{(\rho_0\,\mathrm{c}_0^2)'}{\rho_0}\;+\;\Phi_0'\right)\partial_r a_\ell^m \;-\; \frac{\mathrm{c}_0^2}{r}\partial_r\hat{b}_\ell^m \;-\; \left(\frac{(\rho_0\,\mathrm{c}_0^2)'}{r\,\rho_0}\;-\;\frac{\mathrm{c}_0^2}{r^2}\;+\;\frac{\Phi_0'}{r}\right)\hat{b}_\ell^m \;=\; \frac{f_r}{\rho_0}\,.\end{aligned} \tag{5.55}$$

Taking the derivative of the hydrostatic equilibrium identity (3.14), we have

$$\Phi_0' \;=\; -\frac{\mathrm{p}_0}{\rho_0} \quad\Rightarrow \frac{\Phi_0'}{\mathrm{c}_0^2} \;=\; -\frac{\mathrm{p}_0'}{\mathrm{c}_0^2\rho_0} \;=\; -\frac{1}{\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0} \quad;\quad \Phi_0''\;+\;\rho_0'\,\Phi_0'\;+\;\mathrm{p}_0''\;=\;0\,, \tag{5.56}$$

and using this to simplify the coefficient of a_ℓ^m in (5.55) and dividing both sides from c_0^2, we obtain

$$\begin{aligned} -\,\partial_r^2 a_\ell^m \,+\, \Big(-\,\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,-\, \frac{2(\rho_0\,\mathrm{c}_0^2)'}{\mathrm{c}_0^2\,\rho_0\,r} \,+\, \frac{2}{r^2} \,+\, \frac{2}{r\,\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0}\Big)\, a_\ell^m \,-\, \Big(\frac{2}{r} \,+\, \frac{(\rho_0\,\mathrm{c}_0^2)'}{\mathrm{c}_0^2\,\rho_0}\Big)\partial_r a_\ell^m \\ -\,\frac{1}{r}\partial_r \hat{b}_\ell^m \,+\, \Big(-\frac{(\rho_0\,\mathrm{c}_0^2)'}{r\,\mathrm{c}_0^2\,\rho_0} \,+\, \frac{1}{r^2} \,+\, \frac{1}{r\,\gamma}\frac{\mathrm{p}_0'}{\mathrm{p}_0}\Big)\,\hat{b}_\ell^m \,=\, \frac{f_r}{\mathrm{c}_0^2\,\rho_0}\,. \end{aligned} \tag{5.57}$$

Identification with (4.33) We will compare the system comprising of (5.57) and (5.52) in unknown a_ℓ^m and $\hat{b}_\ell^m$, with system (4.33) given by the main approach which uses directly the vector harmonic basis. The a_ℓ^m and f_ℓ^m are exactly the same between the two sections. On the other hand, using (4.4), the coefficient b_ℓ^m in Section 4 can be related to the $\hat{b}_\ell^m$, by

$$-\sqrt{\ell(\ell+1)}\,b_\ell^m \,=\, \hat{b}_\ell^m \quad , \quad -\sqrt{\ell(\ell+1)}\,g_\ell^m \,=\, [f_h]_\ell^m\,. \tag{5.58}$$

With this identification, we replace $\mathrm{c}_0^2\rho_0$ by $\gamma\mathrm{p}_0$, then the two systems of equations are identical.

6 Computation of the vectorial quantities

In this section, we provide the numerical steps for the computation of the different quantities involved in the vectorial potential with spherical symmetry, derived in Section 4. In particular, to compute V_ℓ from Proposition 6, we need $\tilde{q}_\ell$, $\hat{q}_\ell$ and q_ℓ of (4.44), that depend on the derivative of the coefficients C given in (4.35). Namely, those coefficients involve the derivative of the physical parameters, in particular,

$$\alpha_{\mathrm{c}_0}'\,, \quad \alpha_{\gamma\mathrm{p}_0}'\,, \quad \alpha_{\mathrm{p}_0}' \quad \text{and} \quad \alpha_{\rho_0}'. \tag{6.1}$$

These can be difficult to obtain numerically as, for instance, we loose accuracy if simply using a finite-difference scheme onto the parameters given by the model S. Note that in our implementation, we first represent the models via cubic splines, see Appendix F.

In the following, we investigate how to efficiently calculate the coefficients. For the sake of clarity, we drop the index ℓ that indicates the dependency with mode in the coefficients, and give in Subsection 6.3 two approaches to compute the potential V_ℓ in the interior of the Sun. In Subsection 6.4, we give the computational steps in the atmosphere. From discussion in Appendix D.1, we need to assume the a priori computation of α_{ρ_0}' and α_{c_0}'.

6.1 Relation between scale heights

Let us first give the relation between the scale height associated to p_0 and $\gamma\,\mathrm{p}_0$. From the adiabatic equation of state given in (3.13), we have

$$\frac{(\gamma\mathrm{p}_0)'}{\gamma\mathrm{p}_0} \,=\, \frac{\mathrm{c}_0^{2\prime}\,\rho_0}{\gamma\mathrm{p}_0} \,+\, \frac{\mathrm{c}_0^2\,\rho_0'}{\gamma\mathrm{p}_0} \,=\, \frac{(\mathrm{c}_0^2)'}{\mathrm{c}_0^2} \,+\, \frac{\rho_0'}{\rho_0}\,. \tag{6.2}$$

It means that

$$\boxed{\alpha_{\gamma\mathrm{p}_0} \,=\, -\frac{(\gamma\mathrm{p}_0)'}{\gamma\mathrm{p}_0} \,=\, 2\,\alpha_{\mathrm{c}_0} \,+\, \alpha_{\rho_0}\,.} \tag{6.3}$$

The above identity also gives

$$\frac{\gamma'}{\gamma} + \frac{\mathrm{p}_0'}{\mathrm{p}_0} = -2\alpha_{\mathrm{c}_0} - \alpha_{\rho_0} \qquad \Rightarrow \qquad \frac{\mathrm{p}_0'}{\mathrm{p}_0} = -2\,\alpha_{\mathrm{c}_0} - \alpha_{\rho_0} - \frac{\gamma'}{\gamma} \,=\, -2\,\alpha_{\mathrm{c}_0} - \alpha_{\rho_0} \,+\, \alpha_\gamma\,, \tag{6.4}$$

hence

$$\boxed{\alpha_{\mathrm{p}_0} \,=\, -\frac{\mathrm{p}_0'}{\mathrm{p}_0} \,=\, 2\,\alpha_{\mathrm{c}_0} \,+\, \alpha_{\rho_0} \,-\, \alpha_\gamma\,.} \tag{6.5}$$

We can take the derivatives on both sides of (6.5), we obtain

$$\frac{\mathrm{p}_0''}{\mathrm{p}_0} \,-\, \Big(\frac{\mathrm{p}_0'}{\mathrm{p}_0}\Big)^2 \,=\, -2\,\alpha_{\mathrm{c}_0}' - \alpha_{\rho_0}' \,+\, \alpha_\gamma'\,, \tag{6.6}$$

such that

$$\boxed{\frac{\mathrm{p}_0''}{\mathrm{p}_0} \,=\, -2\,\alpha_{\mathrm{c}_0}' \,+\, \alpha_\gamma' \,-\, \alpha_{\rho_0}' \,+\, (2\,\alpha_{\mathrm{c}_0} \,+\, \alpha_{\rho_0} \,-\, \alpha_\gamma)^2\,.} \tag{6.7}$$

6.2 Relation among parameters under the hydrostatic assumption

We recall the hydrostatic equilibrium identity (3.14) in the interior of the Sun,

$$\mathrm{p}_0' \,+\, \rho_0\,\Phi_0' \,=\, 0\,. \tag{6.8}$$

Equivalently, we have,

$$\Phi_0' \,=\, -\frac{\mathrm{p}_0'}{\rho_0} \,=\, -\frac{\mathrm{p}_0'}{\mathrm{p}_0}\frac{\mathrm{p}_0}{\rho_0} \qquad \Rightarrow \qquad \boxed{\Phi_0' \,=\, \alpha_{\mathrm{p}_0}\frac{\mathrm{c}_0^2}{\gamma}}. \tag{6.9}$$

By applying the derivative to the hydrostatic equilibrium identity (6.8),

$$\rho_0\,\Phi_0'' \,+\, \rho_0'\Phi_0' \,=\, -\mathrm{p}_0'' \quad \Rightarrow \quad \Phi_0'' \,=\, -\frac{\mathrm{p}_0''}{\rho_0} \,-\, \frac{\rho_0'}{\rho_0}\,\Phi_0'\,, \tag{6.10}$$

we have

$$\boxed{\Phi_0'' \,=\, -\frac{\mathrm{p}_0''}{\mathrm{p}_0}\frac{\mathrm{c}_0^2}{\gamma} \,+\, \alpha_{\rho_0}\,\Phi_0'\,.} \tag{6.11}$$

In addition, from the definition of Φ_0 from (1.2) and (4.13),

$$\Phi_0'' \,+\, \frac{2}{r}\Phi_0' \,=\, 4\pi G\,\rho_0 \quad \Leftrightarrow \quad \frac{1}{r^2}\left(r^2\,\Phi_0'\right)' \,=\, 4\,\pi\,G\,\rho_0\,. \tag{6.12}$$

On the other, from (6.10), we can replace Φ_0' by (6.8) and Φ_0'' by (6.12), such that

$$\Phi_0' \,=\, -\frac{\mathrm{p}_0'}{\rho_0}\,; \tag{6.13a}$$

$$\Phi_0'' \,= 4\,\pi G\,\rho_0 \,-\, \frac{2}{r}\Phi_0' \qquad \Rightarrow \qquad \Phi_0'' \,=\, 4\,\pi G\,\rho_0 \,+\, \frac{2}{r}\frac{\mathrm{p}_0'}{\rho_0}\,. \tag{6.13b}$$

Therefore, we have

$$\begin{aligned}\mathrm{p}_0'' \,+\, \rho_0'\,\Phi_0' \,+\, \rho_0\,\Phi_0'' \,&=\, \mathrm{p}_0'' \,-\, \rho_0'\,\frac{\mathrm{p}_0'}{\rho_0} \,+\, \rho_0\,\left(4\,\pi G\,\rho_0 \,+\, \frac{2}{r}\frac{\mathrm{p}_0'}{\rho_0}\right)\\ &=\, \mathrm{p}_0'' \,+\, \left(\frac{2}{r} \,+\, \alpha_{\rho_0}\right)\,\mathrm{p}_0' \,+\, 4\,\pi G\,\rho_0^2\\ &=\, \mathrm{p}_0'' \,+\, (2\,\log r \,-\, \log\rho_0)'\,\mathrm{p}_0' \,+\, 4\,\pi G\,\rho_0^2\,.\end{aligned} \tag{6.14}$$

In the last equality, we have used the definition of $\alpha_{\rho_0} \,=\, -\frac{\rho_0'}{\rho_0} \,=\, -(\log\rho_0)'$. Thus, we obtain

$$\mathrm{p}_0'' \,+\, (2\,\log r \,-\, \log\rho_0)'\,\mathrm{p}_0' \,+\, 4\,\pi G\,\rho_0^2 \,=\, 0 \tag{6.15a}$$

$$\Leftrightarrow \quad \mathrm{p}_0'' \,+\, \left(\frac{2}{r} \,+\, \alpha_{\rho_0}\right)\,\mathrm{p}_0' \,+\, 4\,\pi G\,\rho_0^2 \,=\, 0\,. \tag{6.15b}$$

We now multiply by $e^{2\log r\,-\,\log\rho_0}$,

$$e^{2\log r\,-\,\log\rho_0}\big(\mathrm{p}_0'' \,+\, (2\,\log r \,-\, \log\rho_0)'\,\mathrm{p}_0'\big) + 4\,\pi G\,\rho_0^2\,e^{2\log r\,-\,\log\rho_0} \,=\, 0, \tag{6.16}$$

this gives

$$\big(e^{2\log r\,-\,\log\rho_0}\,\mathrm{p}_0'\big)' \,=\, -4\,\pi G\,\rho_0^2\,e^{2\log r\,-\,\log\rho_0} \tag{6.17}$$

On the other hand, since

$$e^{2\log r\,-\,\log\rho_0} \,=\, \frac{r^2}{\rho_0}\,, \tag{6.18}$$

we obtain the following ODE for p_0,

$$\left(\frac{r^2}{\rho_0}\,\mathrm{p}_0'\right)' \,=\, 4\,\pi G\,\rho_0^2\,\frac{r^2}{\rho_0}\,, \tag{6.19}$$

which simplifies to

$$\boxed{\left(\frac{r^2}{\rho_0}\,\mathrm{p}_0'\right)' \,=\, 4\,\pi G\,\rho_0\,r^2\,.} \tag{6.20}$$

6.3 Computational steps for V_ℓ in the interior

We recall the given quantities from model S are ρ_0, c_0 and γ. Additionally, we also have a choice for attenuation, either constant or from the power law, see Section 2. To compute the potential V_ℓ and the coefficients of the radial ODE (4.42), we give the following steps, which are computed for all positions r.

1. Start from the given background quantities c_0, ρ_0, γ and Γ.
2. Compute p_0 using (3.13): $\mathrm{p}_0 = \mathrm{c}_0^2\,\rho_0\,\gamma^{-1}$.
3. Compute the scale height functions (e.g., using finite-difference formulas or from a cubic spline representation of the models, see Appendix F):

$$\alpha_{\rho_0} \quad \text{and} \quad \alpha_{\mathrm{c}_0}\,, \tag{6.21}$$

and their derivatives,

$$\alpha_{\rho_0}' \quad \text{and} \quad \alpha_{\mathrm{c}_0}'\,. \tag{6.22}$$

4. Compute the derivatives of the attenuation: Γ' and Γ''.
5. The complex frequency given by

$$\sigma \,=\, \omega\,\left(1\,+\,\mathrm{i}\frac{2\Gamma}{\omega}\right)^{1/2}\,, \tag{6.23}$$

and we also need the derivatives of σ:

$$(\sigma^2)' \,=\, \mathrm{i}\,\omega\,(2\Gamma)'\,, \tag{6.24a}$$

$$(\sigma^2)'' \,=\, \mathrm{i}\,\omega\,(2\Gamma)''\,, \tag{6.24b}$$

$$\sigma' \,=\, \frac{1}{2}\frac{(\sigma^2)'}{\sigma}\,, \tag{6.24c}$$

$$\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)' \,=\, 2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0}\,+\,\frac{(\sigma^2)'}{\mathrm{c}_0^2}\,, \tag{6.24d}$$

$$\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)'' \,=\, 2\,\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)'\,\alpha_{\mathrm{c}_0}\,+\,2\,\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)\,\alpha_{\mathrm{c}_0}'\,+\,\frac{(\sigma^2)''}{\mathrm{c}_0^2}\,+\,2\frac{(\sigma^2)'}{\mathrm{c}_0^2}\alpha_{\mathrm{c}_0}\,. \tag{6.24e}$$

Instead of calculating α_γ and its derivative directly from the data of model S, we only assume the inverse scale height and its derivative for c_0 and ρ_0, and exploit the hydrostatic condition and the ODE (4.16) satisfied by Φ_0, and/or the ODE (6.20) for p_0.

6. Compute the weight mass,

$$\mathfrak{M}(r) \,:=\, \int_0^r s^2\,\rho_0(s)\,ds\,. \tag{6.25}$$

7. We use it with (4.15) and (4.16) to obtain Φ_0' and Φ_0'',

$$\Phi_0'(r) \,:=\, 4\pi\,G\,\frac{\mathfrak{M}(r)}{r^2}\,; \tag{6.26a}$$

$$\Phi_0''(r) \,:=\, 4\pi G\,\rho_0(r)\,-\,2\frac{\Phi_0'(r)}{r}\,, \tag{6.26b}$$

8. Compute the derivatives of p_0 using (6.20),

$$\mathrm{p}_0' = -\rho_0(r)\,\Phi_0'(r)\,; \tag{6.27a}$$

$$\mathrm{p}_0'' = -\left(\frac{2}{r} + \alpha_{\rho_0}\right)\mathrm{p}_0' - 4\pi G\,\rho_0^2\,. \tag{6.27b}$$

9. We can now compute the scale heights with the relations

$$\alpha_{\mathrm{p}_0}\left(= -\frac{\mathrm{p}_0'}{\mathrm{p}_0}\right) = \Phi_0'\frac{\rho_0}{\mathrm{p}_0} = \Phi_0'\frac{\gamma}{\mathrm{c}_0^2}\,; \tag{6.28a}$$

$$\alpha_{\gamma\mathrm{p}_0} = 2\alpha_{\mathrm{c}_0} + \alpha_{\rho_0}\,; \tag{6.28b}$$

$$\alpha_\gamma = \alpha_{\gamma\mathrm{p}_0} - \alpha_{\mathrm{p}_0}\,. \tag{6.28c}$$

10. Compute their derivatives

$$\alpha_{\mathrm{p}_0}' = -\frac{\mathrm{p}_0''}{\mathrm{p}_0} + \alpha_{\mathrm{p}_0}^2 \quad = -(2\,r^{-1} + \alpha)\,\alpha_{\mathrm{p}_0}\,4\,\pi\,G\frac{\rho_0\,\gamma}{\mathrm{c}_0^2} + \alpha_{\mathrm{p}_0}^2\,; \tag{6.29a}$$

$$\alpha_{\gamma\mathrm{p}_0}' = 2\alpha_{\mathrm{c}_0}' + \alpha_{\rho_0}'\,. \tag{6.29b}$$

11. Compute

$$\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' \overset{(6.28a)}{=} \left(\frac{\Phi_0'}{\mathrm{c}_0^2}\right)' = \frac{\Phi_0''}{\mathrm{c}_0^2} + 2\frac{\Phi_0'}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0}\,. \tag{6.30}$$

Remaining steps

6. We calculate $r^2C_{22}(r)q(r)$, $r^2C_{22}(r)q(r)$ and $r^2C_{22}\tilde{q}(r)$ by the expression given by Proposition 4.

7. If we work with the conjugate ODE, then $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{g}$ are calculated directly using Proposition 7. The potential $V_\ell(r)$ is given by

$$V_\ell = \frac{1}{4}\,\mathfrak{h}^2 - \frac{1}{2}\,\mathfrak{h}' + \mathfrak{g}\,. \tag{6.31}$$

Remark 13 (Dimensionless of the coefficents of the ODE)**.** *In our current convention, we work with the scaled radius thus dimensionless, and the scaled velocity of unit* s^{-1}*. All of the components of matrix B and C, cf.* (4.29)*, and as a result of this, the coefficients of the radial ODE $\hat{q}$, $\tilde{q}$, q or the scaled version $r^2\,C_{22}(r)\,\hat{q}$, $r^2\,C_{22}(r)\,\tilde{q}$, $r^2\,C_{22}(r)\,q$, are dimensionless. Since r and all of the inverse scale heights $\alpha_\bullet$ are dimensionless, it remains to verify terms such as*

$$\frac{\sigma^2}{\mathrm{c}_0^2}\quad,\quad\frac{\Phi_0''}{\mathrm{c}_0^2}\quad,\quad\frac{\Phi_0'}{\mathrm{c}_0^2}\,. \tag{6.32}$$

The dimensionless of the first term is clear since both the scaled velocity and σ have unit s^{-1}*. Since the scaled radius is dimensionaless, integration or differentiation with respect to this variable does not change units. For this reason, Φ_0' and Φ_0'' have the same unit as Φ_0. To determine the unit of Φ_0, we can consider* (4.14)*, from which it is defined. In this way, Φ_0 has the same unit as the right-hand-side, $4\pi G\rho_0$. Using the value given in* (6.33h)*, G is* $6.674\,08\times10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$*, on the other hand ρ_0 in the model $\mathcal{S}$ is given in* $\mathrm{g\,cm^{-3}}$*. Thus their product is in* s^{-2}*. Thus Φ_0, Φ_0' and Φ_0'' in the current convention take on units* s^{-2}*. As a result of this, all of the expression in* (6.32) *are dimensionless.* △

6.4 Computational steps for V_ℓ in the atmosphere

In the atmosphere with model `AtmoCAI`, the steps are simpler as the sound speed, adiabatic coefficient and density scale height are constant. In addition, we can readily obtain the value of the parameters, extracted from the end of model S, such that

$$\begin{aligned}
r_a &= 1.000\,712\,6\,, && (6.33a)\\
\rho_0(r_a) &= 3.062\,97 \times 10^{-9}\,\mathrm{g\,cm^{-3}}\,, && (6.33b)\\
\alpha_{\rho_0} &= 6636.41\,, && (6.33c)\\
\gamma(r_a) &= 1.640\,921\,1\,, && (6.33d)\\
\mathrm{p}_0(r_a) &= 9.455\,763\,9 \times 10^{1}\,\mathrm{Pa} \,=\, 9.455\,763\,9 \times 10^{2}\,\mathrm{dyn/cm^2}\,, && (6.33e)\\
\mathrm{c}_0(r_a) &= \frac{6.8569 \times 10^{5}}{R_\odot} \,=\, 9.8588 \times 10^{-6}\,\mathrm{s^{-1}}\,, && (6.33f)\\
G &= 6.674\,08 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}} && (6.33g)\\
&= 6.674\,08 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}\,. && (6.33h)
\end{aligned}$$

We have the following steps for the computation of the potential in the atmosphere.

1. Compute the derivative of the background gravitational potential $\Phi_0'(r)$,

$$\begin{aligned}
\mathfrak{m} &= 4\pi \int_0^{r_a} s^2 \rho_0(s)\,ds \,+\, 4\pi\,\rho_0(r_a)\frac{(\alpha_{\rho_0}\,r_a)^2 \,+\, 2\,r_a\,\alpha_{\rho_0} + 2}{\alpha^3}\,,\\
\Phi_0'(r) &= \frac{G}{r^2}\,\mathfrak{m} \,-\, 4\pi\,G\,\rho_0(r_a)\;\frac{e^{-\alpha_{\rho_0}(r-r_a)}}{r^2}\,\frac{(\alpha_{\rho_0}\,r)^2 \,+\, 2\,r\,\alpha_{\rho_0} \,+\, 2}{\alpha_{\rho_0}^3}\,,\\
\Phi_0''(r) &\overset{(4.16)}{=} 4\pi\,G\,\rho_0 \,-\, \frac{2}{r}\,\Phi_0'(r)\,.
\end{aligned} \qquad (6.34)$$

2. We calculate $r^2 C_{22}(r) q(r)$, $r^2 C_{22}(r) q(r)$ and $r^2 C_{22}\tilde{q}(r)$ by the expression given by Proposition 5.

3. If we work with the conjugate ODE, then $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{g}$ are calculated directly using Proposition 8. The potential $V_\ell(r)$ is given by

$$V_\ell \,=\, \frac{1}{4}\,\mathfrak{h}^2 \,-\, \frac{1}{2}\,\mathfrak{h}' \,+\, \mathfrak{g}\,. \qquad (6.35)$$

7 Analysis of the modal ODE: indicial roots

In Section 4, we have established that the Galbrun's equation (1.1) for the model parameters S+AtmoCAI reduces to solving (4.42), which we recall for convenience, this is from Proposition 6:

$$\Big(\hat{q}_\ell(r)\,\partial_r^2 \,+\, q_\ell(r)\,\partial_r \,+\, \tilde{q}_\ell(r)\Big) a_\ell^m \,=\, \mathfrak{f}_\ell^m(r)\,, \qquad (7.1)$$

for the radial coefficient a_ℓ^m, with q_ℓ, $\tilde{q}_\ell$ and $\hat{q}_\ell$ given in (4.44). This equation is equivalent to the *conjugate radial ODE*

$$\big(-\partial_r^2 \,+\, V_\ell(r)\big)\,\tilde{a}_\ell^m \,=\, -e^{-\frac{1}{2}\int_0^r \mathfrak{h}_\ell(r)}\,\frac{\mathfrak{f}_\ell^m(r)}{\hat{q}_\ell(r)}\,, \qquad (7.2)$$

where the modal potential V_ℓ is given in terms of function $\mathfrak{h}_\ell(r)$ and $\mathfrak{g}_\ell(r)$ as

$$V_\ell(r) \,=\, \frac{1}{4}\mathfrak{h}_\ell^2(r) \,-\, \frac{1}{2}\partial_r \mathfrak{h}_\ell(r) \,+\, \mathfrak{g}_\ell(r)\,, \qquad (7.3)$$

with

$$\mathfrak{h}_\ell(r) \,=\, -\frac{q_\ell(r)}{\hat{q}_\ell(r)}\,, \qquad \mathfrak{g}_\ell(r) \,=\, -\frac{\tilde{q}_\ell(r)}{\hat{q}_\ell(r)}\,,$$

given in Proposition 3. The coefficients $\tilde{a}_\ell^m = \tilde{a}_\ell$ are called the *conjugate radial coefficients* and are related to the original radial coefficients a_ℓ^m by

$$a_\ell^m(r) \,=\, \tilde{a}_\ell^m(r)\,e^{\frac{1}{2}\int_0^r \mathfrak{h}_\ell(r)}\,. \qquad (7.4)$$

Due to the independence of m, we will drop the index m from all of the quantities.

In this section, we study the local behavior of the coefficients of (7.1) and (7.2). We show that the coefficients of (7.1) contain at most two singular points, and both of which are regular singular. When there is attenuation, $\Gamma > 0$, V_ℓ only has one singularity at $r = 0$. Without attenuation, $\Gamma = 0$, V_ℓ has two singularities: at $r = 0$ and at a point called $r^{\star}_{\mathrm{i},\omega,\ell}$. Then, the asymptotic behavior at infinity is studied in Section 8.

Let us first note that, as we study the regularity of the coefficients of the ODE, it suffices to consider the version of (7.1) with zero right-hand side,

$$\hat{q}(r)\,\partial_r^2 a \,+\, q(r)\,\partial_r a \,+\, \tilde{q}(r)\,a \,=\, 0\,. \tag{7.5}$$

In fact, due to the form of the coefficients, it is simpler to study the regularity after multiplying both sides of (7.5) by $r^2 C_{22}$, we refer to Subsection 6.3 for more details (see (D.16) and (D.23)). We have

$$r^2\,C_{22}(r)\,\hat{q}(r)\,\partial_r^2 a \,+\, r^2\,C_{22}\,q(r)\,\partial_r a \,+\, r^2\,C_{22}\,\tilde{q}(r)\,a \,=\, 0\,, \tag{7.6}$$

where C_{22} is given in (4.35).

We first recall the classification of the singular points from [13, Theorem 4 p. 164] or [32, Section 1.1].

Definition 2 (Classification of the singular points)**.** *A point $r = r_0$ is a singularity of finite order of (7.6), if it is a pole of finite order of $\frac{q(r)}{\hat{q}(r)}$ or $\frac{\tilde{q}(r)}{\hat{q}(r)}$. In particular, $r = r_0$ is a regular singular point if $\frac{q(r)}{\hat{q}(r)}$ has a pole at $r = r_0$ of at most first order and $\frac{\tilde{q}(r)}{\hat{q}(r)}$ has a pole at $r = r_0$ of at most second order. In that case, we can define the* indicial roots *or* characteristics exponent *(denoted by λ) as the roots of the* indicial equation

$$\lambda\,(\lambda \,-\, 1) \,+\, \eta\,\lambda \,+\, \tilde{\eta} \,=\, 0\,, \tag{7.7}$$

whose coefficients are given by

$$\eta \,:=\, \lim_{r\to r_0}\,(r \,-\, r_0)\,\frac{q(r)}{\hat{q}(r)}\,, \qquad \tilde{\eta} \,:=\, \lim_{r\to r_0}\,(r \,-\, r_0)^2\,\frac{\tilde{q}(r)}{\hat{q}(r)}\,. \tag{7.8}$$

In the following, we distinguish the case of the interior of the sun and of the atmosphere, with ($\Gamma \neq 0$) and without attenuation ($\Gamma = 0$). Our main results are given in Propositions 10, 11, 14 and 15, and summarized in Table 1.

			Indicial exponent λ for the	
Singular point			Radial ODE $\lambda^{\pm}$	Conjugate radial ODE $\tilde{\lambda}^{\pm} = \lambda^{\pm} - \frac{1}{2}\eta$, cf. (7.125)
0	$\Gamma \geq 0$	$\ell = 0$	-2 , 1	$\eta = 2$
		$\ell > 0$	$-\ell - 2$, $\ell - 1$	$\eta = 4$
$r^{\star}_{\mathrm{i},\omega,\ell}$, cf. (7.19)	$\Gamma = 0$	$0 < \ell \leq \ell^{\star}_{\omega}$, cf. (7.20)	0, $r^{\star}_{\mathrm{i},\omega,\ell} + 1$	$\eta = -r^{\star}_{\mathrm{i},\omega,\ell}$
$r^{\star}_{\mathrm{a1},\omega}$, cf. (7.60)		$\omega < \omega^{\star}_{a1}$, cf. (7.83) and Assumption 5	0	η
$r^{\star}_{\mathrm{a2},\omega,\ell}$, cf. (7.65)		$\ell > \ell^{\star}_{a,\omega}$, cf. (7.71)	0 , 2	$\eta = -1$

Table 1: Sets of singular points for the modal radial equation (7.1) and conjugate one (7.2). This table summarizes the results obtained from Propositions 10, 11, 14 and 15. The indexes a and i of the singular points $r^{\star}$ indicate if it is located in the interior of the Sun or in the atmosphere, respectively. The indexes ω and ℓ indicate the dependency of the points with frequency and mode, respectively.

7.1 Indicial roots analysis at $\ell = 0$

Following Remarks 9 and 10, due to the simplicity of the equation at $\ell = 0$, cf. (4.32), we can study directly,

$$-\partial_r^2\,a_0 \,+\, \left(2\alpha_{\mathrm{p}_0} \,-\, \frac{2}{r}\right)\,\partial_r\,a_0 \,+\, \left(-\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,+\, 2\frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,+\, \frac{2}{r^2} \,-\, 2\frac{\alpha_{\mathrm{p}_0}}{r\gamma}\right)\,a_0 \,=\, 0\,. \tag{7.9}$$

It leads at the fact that $r = 0$ is the only singular point on $[0, \infty)$, with indicial equation,

$$s(s-1) + 2s - 2 = 0\,. \tag{7.10}$$

Consequently, the associated indicial exponents are

$$-2 \quad \text{and} \quad 1\,, \qquad\qquad \text{indicial exponents at } \ell = 0. \tag{7.11}$$

7.2 Indicial analysis in the interior of the Sun for positive ℓ

We consider (7.6) in the interior of the Sun, that is for $r \leq r_a$, where the physical parameters are given by the model S of [12], that we pictured in Figures 1 and 2. From the expression of the coefficients of Subsection 6.3, (D.16) and (D.23), we note that the only possible singularities are at $r = 0$, at the root of $r^2 C_{22} \hat{q}$ and the roots of C_{22}.

$$\frac{\sigma^2}{c_0^2} r^2\, \partial_r^2 a \,+\, r^2\, C_{22}(r)\, q(r)\, \partial_r a \,+\, r^2\, C_{22}(r)\, \tilde{q}(r)\, a \,=\, 0\,, \tag{7.12}$$

or

$$\boxed{r^2\, \partial_r^2 a \,+\, \frac{c_0^2}{\sigma^2}\, r^2\, C_{22}(r)\, q(r)\, \partial_r a \,+\, \frac{c_0^2}{\sigma^2}\, r^2\, C_{22}(r)\, \tilde{q}(r)\, a \,=\, 0\,.} \tag{7.13}$$

From their derivation in Subsection 6.3, (D.13) and (D.16), we have

$$C_{22}(r) \,=\, -\frac{\sigma^2}{c_0^2} \,+\, \frac{\ell(\ell+1)}{r^2} \qquad \Rightarrow \qquad r^2\, C_{22}(r)\, \hat{q}(r) \,=\, -r^2\, C_{22}(r) \,+\, \ell(\ell+1) \,=\, \frac{\sigma^2}{c_0^2} r^2\,. \tag{7.14}$$

Therefore, from the discussion of regular singularity and Frobenius method, we need to investigate the regularity of

$$\frac{c_0^2}{\sigma^2}\, C_{22}\, q \qquad \text{and} \qquad \frac{c_0^2}{\sigma^2}\, r^2\, C_{22}\, \tilde{q}\,. \tag{7.15}$$

We recall the explicit expression of coefficients of ODE (7.12), given in Proposition 4.

$$\frac{r\, C_{22}\, q(r)}{k_0^2} \,=\, -\alpha_{\gamma p_0}\, r \,+\, 2 \,-\, \frac{\ell(\ell+1)}{k_0^2}\, \frac{2\,\alpha_{c_0}\, r \,+\, \mathrm{i}\,\omega\, \frac{(2\Gamma)'}{c_0^2}\, \frac{1}{k_0^2}\, r \,+\, 2}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,. \tag{7.16}$$

and

$$\begin{aligned}
\frac{r^2\, C_{22}\, \tilde{q}(r)}{k_0^2} \,=\, &\left(k_0^2 \,-\, \frac{\Phi_0''}{c_0^2}\right) r^2 \,+\, 2\, r \left(\frac{\Phi_0'}{c_0^2} \,-\, \alpha_{\rho_0} \,-\, 2\alpha_{c_0}\right) \,-\, 2 \,-\, \ell(\ell+1) \\
&+\, \frac{\ell(\ell+1)}{k_0^2}\, \frac{\Phi_0'}{c_0^2} \left(\alpha_{\rho_0} \,-\, \frac{\Phi_0'}{c_0^2}\right) \\
&-\, \frac{\ell(\ell+1)}{k_0^2} \left(2 \,-\, \frac{\Phi_0'}{c_0^2}\, r\right) \frac{2\,\alpha_{c_0}\, r \,+\, 2 \,+\, \mathrm{i}\, \frac{\omega}{k_0^2}\, \frac{(2\Gamma)'}{c_0^2} r}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,.
\end{aligned} \tag{7.17}$$

If we assume that the background quantities are regular enough at $r = 0$ (see, e.g., assumption (7.26)), from the above expressions of the coefficients, we note that the only possible singular points in the coefficients of (7.12) are $r = 0$ and the zeros of the algebraic equation on $r \geq 0$,

$$C_{22}(r) \,=\, 0 \quad \Leftrightarrow \quad r \,=\, \frac{c_0(r)}{\omega}\, \sqrt{\ell(\ell+1)} \quad \Leftrightarrow \quad \frac{r}{c_0(r)} \,=\, \frac{\sqrt{\ell(\ell+1)}}{\omega}\,. \tag{7.18}$$

– When $\Gamma \neq 0$, ω is complex and the roots of (7.18) are complex. In this case, since we consider the ODE (7.6) on $\mathbb{R}^+$, the only singular point of the coefficients of (7.6) is at $r = 0$.

– When $\Gamma = 0$, (7.18) has real roots. Model `S+Atmo` satisfies assumption Assumption 3,

$$r \mapsto \frac{r}{c_0(r)} \quad \text{is increasing on } [0, r_a]\,,$$

as illustrated in Figure 2b. Under this assumption, equation (7.18) has at most one zero. For each ℓ and ω, denote by $r^\star_{\mathrm{i},\omega,\ell}$ the unique zero to (7.18), if it exists,

$$\boxed{r^\star_{\mathrm{i},\omega,\ell} \,=\, \frac{c_0(r^\star_{\mathrm{i},\omega,\ell})}{\omega}\sqrt{\ell(\ell+1)} \quad\Leftrightarrow\quad \frac{r^\star_{\mathrm{i},\omega,\ell}}{c_0(r^\star_{\mathrm{i},\omega,\ell})} \,=\, \frac{\sqrt{\ell(\ell+1)}}{\omega}}\,. \tag{7.19}$$

The existence is discussed in the Proposition 9.

We define

$$\boxed{\ell^\star_\omega \,:=\, -\frac{1}{2} + \sqrt{\frac{\omega^2\, r_a^2}{c_0(r_a)^2} + \frac{1}{4}}.} \tag{7.20}$$

Proposition 9. *Under Assumption 3 and the continuity of $r \mapsto c_0(r)$, when $\Gamma = 0$, at each fixed $\omega > 0$, we have the following equivalence*

$$\textit{Equation (7.18) has a unique zero on } (0, r_a) \qquad\Leftrightarrow\qquad \ell \leq \ell^\star_\omega\,. \tag{7.21}$$

Proof. **Statement** ($\Rightarrow$) Suppose equation (7.18) has a zero on $(0, r_a)$, then by the above discussion it is unique and is denoted by $r^\star_{\mathrm{i},\omega,\ell}$. We have

$$r^\star_{\mathrm{i},\omega,\ell} \,=\, \frac{c_0(r^\star_{\mathrm{i},\omega,\ell})}{\omega}\sqrt{\ell(\ell+1)} \qquad\Leftrightarrow\qquad \frac{r^\star_{\mathrm{i},\omega,\ell}}{c_0(r^\star_{\mathrm{i},\omega,\ell})} \,=\, \frac{\sqrt{\ell(\ell+1)}}{\omega}\,.$$

Since $r^\star_{\mathrm{i},\omega,\ell} \leq r_a$, and $r \mapsto \frac{r}{c_0(r)}$ is a strictly increasing function, this implies

$$\frac{\sqrt{\ell(\ell+1)}}{\omega} \,=\, \frac{r^\star_{\mathrm{i},\omega,\ell}}{c_0(r^\star_{\mathrm{i},\omega,\ell})} \,\leq\, \frac{r_a}{c_0(r_a)}\,. \tag{7.22}$$

This leads to

$$\frac{r_a}{c_0(r_a)} \,\geq\, \frac{\sqrt{\ell(\ell+1)}}{\omega} \quad\Leftrightarrow\quad \frac{r_a^2}{c_0^2(r_a)}\omega^2 + \frac{1}{4} \,\geq\, \left(\ell + \frac{1}{2}\right)^2 \quad\Leftrightarrow\quad \ell^\star_\omega \,\geq\, \ell\,. \tag{7.23}$$

Statement ($\Leftarrow$) Suppose $\ell \leq \ell^\star_\omega$. It suffices to consider the existence statement. We consider function

$$f \;:\; r \;\mapsto\; r - \frac{c_0(r)}{\omega}\sqrt{\ell(\ell+1)}\,. \tag{7.24}$$

Under the current assumption, from the equivalence in (7.23), we have readily that $f(r_a) \geq 0$. On the other hand, since $c_0 > 0$, $f(0) < 0$. By the continuity of $r \mapsto f(r)$, this implies that $f(r) = 0$ has at least one zero on $(0, r_a)$. □

Remark 14. *Using the solar parameters given by the model* `S`*, the values can be explicitly obtained and we have*

$$\frac{r_a}{c_0(r_a)} \simeq 1.015 \times 10^5\mathrm{s} \qquad \textit{for model } \texttt{S}. \tag{7.25}$$

In solar applications, the frequency is usually given in mHz *and we study the first hundreds of modes. For instance, at* 1 mHz*, $\ell^\star_\omega = 637$. We further illustrate the position of the singularity in Subsection 7.5.1. On the other hand, this singularity does not exist when we consider attenuation, which is the case in applications.* △

We next verify that the points $r = 0$ and $r^\star_{\mathrm{i},\omega,\ell}$, with $\ell > 0$ defined in (7.18) are indeed regular singularities and calculate the corresponding indicial exponents.

Proposition 10 (Singularities at $r = 0$). *We work under the assumption,*

$$\Gamma,\ \mathrm{c}_0,\ \alpha_{\rho_0},\ \alpha_{\mathrm{c}_0},\ \alpha_\gamma \textit{ and their derivatives are regular on } [0,\infty). \tag{7.26}$$

1. *The origin $r = 0$ is a regular singular point of the ODE (7.5) on $r \le r_a$, with indicial equation*

$$s\,(s\,-\,1)\,+\,\eta_0\,s\,+\,\tilde{\eta}_0\,=\,0 \tag{7.27}$$

with

$$\eta_0\,=\,\lim_{r\to 0} r\,\frac{\mathrm{c}_0^2}{\sigma^2}\,C_{22}(r)\,q(r)\,=\,\begin{cases} 2 & \textit{for } \ell\,=\,0\,,\\ 4 & \textit{for } \ell\,>\,0\,,\end{cases} \tag{7.28}$$

and

$$\tilde{\eta}_0\,=\,\lim_{r\to 0} r^2\,\frac{\mathrm{c}_0^2}{\sigma^2}\,C_{22}(r)\,\tilde{q}(r)\,=\,\begin{cases} -2\,, & \ell = 0\\ 2\,-\,\ell(\ell+1) & \ell > 0\,.\end{cases} \tag{7.29}$$

The indicial exponents are given by

$$\lambda_0^+\,=\,\ell\,-\,1\,,\qquad \lambda_0^-\,=\,-\ell\,-\,2\,,\qquad \textit{for } \ell > 0\,, \tag{7.30}$$

and

$$\lambda_0^-\,=\,-2\,,\qquad \lambda_0^+\,=\,1\,,\qquad \textit{for } \ell = 0. \tag{7.31}$$

2. *When $\ell = 0$, in the cases with or without attenuation, $r = 0$ is the only singular point of (7.5) on $0 \le r \le r_a$, with indicial equation (7.38b).*
3. *When $\Gamma > 0$, $r = 0$ is the only real regular singular point on (7.5) on $[0, r_a]$ for all ℓ.*

Proof. It suffices to consider the case for $\ell > 0$. From Proposition 4, we have $r\,C_{22}\,q$ with

$$r\,C_{22}(r)\,q(r)\,=\,4\frac{\sigma^2(0)}{\mathrm{c}_0^2(0)}\,+\,\mathsf{O}(r)\,,\qquad r\to 0\,,\qquad \ell > 0\,, \tag{7.32}$$

and

$$r\,C_{22}(r)\,q(r)\,=\,2\frac{\sigma^2(0)}{\mathrm{c}_0^2(0)}\,+\,\mathsf{O}(r)\,,\qquad r\to 0\,,\qquad \ell > 0\,. \tag{7.33}$$

From here, we obtain readily the value of η_0 in (7.28).

Also from Proposition 4, we have $r^2\,C_{22}\,\tilde{q}$ are regular at $r = 0$ and for $\ell > 0$,

$$\frac{r^2\,C_{22}(r)\,\tilde{q}(r)}{k_0^2}\,=\,2-\ell(\ell+1)\,+\,\frac{\ell(\ell+1)}{k_0^2}\left(-\left(\frac{\Phi_0'}{\mathrm{c}_0^2}\right)^2 + \alpha_{\rho_0}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\right)\,+\,\mathsf{O}(r)\,,\quad r\to 0\,. \tag{7.34}$$

The zero-th coefficient $\tilde{\eta}_0$ is then

$$\tilde{\eta}_0\,=\,2\,-\,\ell(\ell+1)\,+\,\frac{\ell(\ell+1)}{k_0^2}\left(-\left(\frac{\Phi_0'}{\mathrm{c}_0^2}\right)^2 + \alpha_{\rho_0}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\right)\,. \tag{7.35}$$

It remains to calculate $\Phi_0'(0)$. From (4.15), we have

$$\Phi'(r)\,=\,4\pi G\frac{\mathfrak{M}(r)}{r^2}\,,\qquad \mathfrak{M}(r)\,=\,\int_0^r s^2\,\rho_0(s)\,ds\,.$$

Using l'Hopital's rule, we obtain

$$\Phi_0'(0)\,=\,4\pi G\lim_{r\to 0}\frac{\mathfrak{M}'(r)}{2r}\,=\,4\pi G\lim_{r\to 0}\frac{r^2\,\rho_0(r)}{2r}\,=\,0\,. \tag{7.36}$$

Consequently, (7.35) simplifies to

$$\tilde{\eta}_0\,=\,2\,-\,\ell(\ell+1)\,. \tag{7.37}$$

Consequently, the indicial equation (7.27) at $r = 0$ of ODE (7.5) takes the following explicit form

$$s^2 + 3\,s + 2 - \ell(\ell+1) = 0 \qquad \ell > 0 \tag{7.38a}$$

$$s^2 + s - 2 = 0\,, \qquad \ell = 0\,. \tag{7.38b}$$

For $\ell > 0$, the discriminant of the quadratic form (7.27) is then

$$\Delta = 3^2 - 4\tilde{\eta}_0 = 1 + 4\,\ell(\ell+1) = 4(\ell + \tfrac{1}{2})^2\,.$$

Thus the indicial exponents at $r = 0$ are given by (7.30):

$$\lambda_0^{\pm} := -\frac{3}{2} \pm (\ell + \frac{1}{2}) \quad , \quad \ell > 0\,. \tag{7.39}$$

$\square$

Remark 15. *From the above calculation, we can also obtain* $\Phi_0''(0)$*,*

$$\frac{\Phi'(r)}{r} = 4\pi G \frac{\mathfrak{M}(r)}{r^3} \quad \Rightarrow \quad \lim_{r\to 0} \frac{\Phi_0'}{r} = 4\pi G \lim_{r\to 0} \frac{\mathfrak{M}'(r)}{3r^2} = \frac{4}{3}\pi G\,\rho_0(0)\,. \tag{7.40}$$

From its definition given by the ODE (4.16),

$$\Phi_0''(0) = 4\pi G\rho_0(0) - 2\,\frac{4}{3}\pi G\rho_0(0) = \frac{4}{3}\pi G\rho_0(0)\,. \tag{7.41}$$

$\triangle$

Remark 16. *The indicial roots near the center* ($r = 0$) *have been derived in [35] under the hypothesis that* ρ *and* c *tend to a constant value while* $\Phi_0' \sim 0$*,* $N^2 \sim 0$ *and* $N^2/\Phi_0' \sim 0$*.* N^2 *is the buoyancy frequency defined by (5.17). The last condition implies that the background is adiabatic near the center. In this case, the homogeneous system obtained in Remark 12 can be written as*

$$r\,\partial_r \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{7.42}$$

where $y_1 = \dfrac{d_l^m}{r}$*,* $y_2 = \dfrac{e_l^m}{r}$ *and*

$$B = \begin{pmatrix} -3 & \dfrac{\ell(\ell+1)}{c_1\sigma^2} \\ c_1\sigma^2 & -2 \end{pmatrix}. \tag{7.43}$$

The indicial roots are the singular values of the matrix B *and are thus* $\ell - 2$ *and* $-\ell - 3$ *for* y_1 *and thus* $\ell - 1$ *and* $-\ell - 2$ *for* ξ_r *as obtained in Proposition 10. However the derivation from [35] does not hold for* $\ell = 0$*.*

Proposition 11 (Singularities apart from 0 in the interior)**.** *Under the assumption (7.26), we have*

1. *For* $\Gamma = 0$ *and a given* $\omega > 0$*, for* $0 < \ell \leq \ell_\omega^\star$ *defined in (7.20), in addition to* $r = 0$*, the ODE (7.5) on* $r \leq r_a$ *also has a regular singular point at* $r = r^\star_{\mathrm{i},\omega,\ell}$ *defined in (7.19) with the indicial root equation*

$$s(s-1) + \eta\, s + 0 = 0\,, \tag{7.44}$$

where

$$\eta := \lim_{r\to r^\star_{\mathrm{i},\omega,\ell}} (r - r^\star_{\mathrm{i},\omega,\ell})\,\frac{c_0^2}{\omega^2}\,C_{22}\,q = -r^\star_{\mathrm{i},\omega,\ell}\,. \tag{7.45}$$

The indicial exponents associated to $r = r^\star_{\mathrm{i},\omega,\ell}$ *are*

$$s = 0 \qquad and \qquad s = -\eta + 1 = r^\star_{\mathrm{i},\omega,\ell} + 1 > 0\,. \tag{7.46}$$

2. *For* $\Gamma = 0$ *and* $\ell > \ell_\omega^\star$*, on* $[0, r_a]$*, (7.5) is singular only at* $r = 0$*.*

Proof. Since $\frac{c_0^2(r)}{\omega^2} C_{22}\tilde{q}(r)$ only has a pole of rank 1 at $r = r^\star_{i,\omega,\ell}$, we have

$$\lim_{r \to r^\star_{i,\omega,\ell}} \left(r - r^\star_{i,\omega,\ell}\right)^2 \frac{c_0^2(r)}{\omega^2} C_{22}(r)\,\tilde{q}(r) = 0\,. \tag{7.47}$$

Next, we consider (4.53) for $\frac{c_0^2(r)}{\omega^2} C_{22}\,q(r)$, which simplifies to

$$\frac{c_0^2(r)}{\sigma^2} C_{22}(r)\,q(r) \overset{\Gamma=0}{=} -\alpha_{\gamma p_0} + \frac{2}{r} - \ell(\ell+1)\frac{c_0^2(r)}{\omega^2} \frac{2\,(\alpha_{c_0} r + 1)}{\left(r - \sqrt{\ell(\ell+1)}\,\frac{c_0}{\omega}\right)\left(r + \sqrt{\ell(\ell+1)}\,\frac{c_0}{\omega}\right)}\,. \tag{7.48}$$

Thus for $\Gamma = 0$, we have

$$\begin{aligned}\left(r - r^\star_{i,\omega,\ell}\right)\frac{c_0^2(r)}{\omega^2} C_{22}(r)\,q(r) &= \left(-\alpha_{\gamma p_0} + \frac{2}{r}\right)\left(r - r^\star_{i,\omega,\ell}\right) \\ &\quad - \ell(\ell+1)\frac{c_0^2(r)}{\omega^2}\frac{2\,(\alpha_{c_0} r + 1)}{r + \sqrt{\ell(\ell+1)}\,\frac{c_0(r)}{\omega}}\,\frac{r - r^\star_{i,\omega,\ell}}{r - \sqrt{\ell(\ell+1)}\,\frac{c_0(r)}{\omega}}\,.\end{aligned} \tag{7.49}$$

The first term will vanish at $r = r^\star_{i,\omega,\ell}$, hence it remains to consider the limit of the second term.

Using the definition of $r^\star_{i,\omega,\ell}$, i.e. $r^\star_{i,\omega,\ell} = \sqrt{\ell(\ell+1)}\frac{c_0(r^\star_{i,\omega,\ell})}{\omega}$, we have

$$\lim_{r \to r^\star_{i,\omega,\ell}} r + \sqrt{\ell(\ell+1)}\,\frac{c_0(r)}{\omega} = 2\,r^\star_{i,\omega,\ell}\,, \tag{7.50}$$

and

$$\begin{aligned}\lim_{r \to r^\star_{i,\omega,\ell}} \frac{r - r^\star_{i,\omega,\ell}}{r - \sqrt{\ell(\ell+1)}\frac{c_0(r)}{\sigma}} &= \frac{1}{1 - \sqrt{\ell(\ell+1)}\,\frac{c_0'(r^\star_{i,\omega,\ell})}{\omega}} = \frac{1}{1 + \sqrt{\ell(\ell+1)}\,\frac{c_0(r^\star_{i,\omega,\ell})}{\omega}\,\alpha_{c_0}(r^\star_{i,\omega,\ell})} \\ &= \frac{1}{1 + r^\star_{i,\omega,\ell}\,\alpha_{c_0}(r^\star_{i,\omega,\ell})} > 0 \quad \text{due to (7.53)}\,.\end{aligned} \tag{7.51}$$

The last inequality follows from assumption Assumption 3 and Figure 2b,

$$r \mapsto \frac{r}{c_0(r)} \quad \text{is strictly increasing on } [0\,,\, r_a]\,. \tag{7.52}$$

This means that

$$\left(\frac{r}{c_0}\right)' = \frac{1}{c_0} + \frac{r}{c_0}\alpha_{c_0} > 0 \quad \Rightarrow \quad 1 + r\,\alpha_{c_0}(r) > 0. \tag{7.53}$$

As a result of this, we have

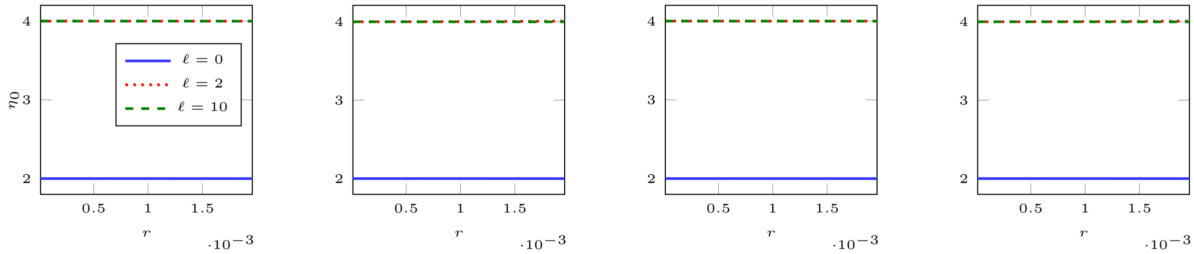
$$\begin{aligned}&\lim_{r \to r^\star_{i,\omega,\ell}} \left(r - r^\star_{i,\omega,\ell}\right)\frac{c_0^2(r)}{\sigma^2} C_{22}(r)\,q(r) = \\ &- (r^\star_{i,\omega,\ell})^2\,\frac{2(r^\star_{i,\omega,\ell}\,\alpha_{c_0}(r^\star_{i,\omega,\ell}) + 1)}{2\,r^\star_{i,\omega,\ell}}\,\frac{1}{1 + r^\star_{i,\omega,\ell}\,\alpha_{c_0}(r^\star_{i,\omega,\ell})} = -\,r^\star_{i,\omega,\ell}\,.\end{aligned} \tag{7.54}$$

□

Numerical validation We have obtained in Proposition 10 the indicial exponents at the origin, with the corresponding values of η_0 and $\tilde{\eta}_0$ for all modes. These coefficients are also defined in terms of the physical parameters (velocity, density) from Propositions 10 and 11, in (7.28) and (7.29). Here, we want to see if the limits are verified numerically, that is, if we have:

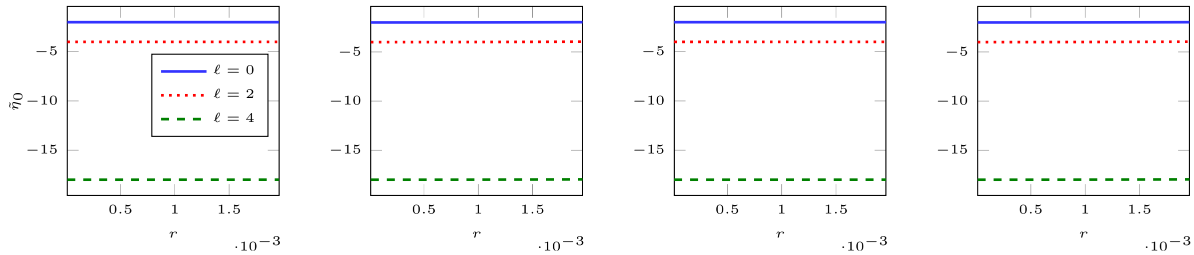
$$\eta_0 = \lim_{r\to 0} r\,\frac{c_0^2}{\sigma^2} C_{22}\,q(r) = \begin{cases} 2 & \text{for } \ell = 0 \\ 4 & \text{for } \ell > 0 \end{cases}, \tag{7.55a}$$

$$\tilde{\eta}_0 = \lim_{r\to 0} r^2\,\frac{c_0^2}{\sigma^2} C_{22}\,\tilde{q}(r) = \begin{cases} -2 & \text{for } \ell = 0 \\ 2 - \ell(\ell+1) & \text{for } \ell > 0 \end{cases}. \tag{7.55b}$$



(a) 2 mHz, $\Gamma = 0$. (b) 10 mHz, $\Gamma = 0$. (c) 2 mHz, $\Gamma/(2\pi) = 20$ µHz. (d) 10 mHz, $\Gamma/(2\pi) = 20$ µHz.

Figure 4: Numerical validation of the limit of η_0 given by (7.55a) at frequency 2 and 10 mHz, with and without attenuation. The numerical computation uses the expression with the limit in (7.28), and follows the step of Section 6.



(a) 2 mHz, $\Gamma = 0$. (b) 10 mHz, $\Gamma = 0$. (c) 2 mHz, $\Gamma/(2\pi) = 20$ µHz. (d) 10 mHz, $\Gamma/(2\pi) = 20$ µHz.

Figure 5: Numerical validation of the limit of $\tilde{\eta}_0$ given by (7.55b) at frequency 2 and 10 mHz, with and without attenuation. The numerical computation uses the expression with the limit in (7.29), and follows the step of Section 6.

In Figures 4 and 5, we respectively compute η_0 and $\tilde{\eta}_0$ using the solar parameters (velocity, density, etc.) and the approach of Section 6. Therefore we evaluate the first parts in (7.55) to see if the limits are retrieved numerically. We plot for different frequencies, and in the absence ($\Gamma = 0$) or presence of attenuation.

We observe in the figures that the limits at the origin are perfectly respected, even when the radius r reaches 2×10^{-3}. Both η_0 and $\tilde{\eta}_0$ are constant, and we retrieve the values expected, according to (7.55). This serves to validate further our analysis.

7.3 Indicial analysis in the atmosphere

We next consider the regularity of the coefficients of the scaled radial ODE (7.6) for $r \geq r_a$,

$$r^2\, C_{22}\, \hat{q}(r)\, \partial_r^2 a \,+\, r^2\, C_{22}\, q(r)\, \partial_r a \,+\, r^2\, C_{22}\, \tilde{q}(r)\, a \,=\, 0\,.$$

Let us first note that c_0, γ and α_{ρ_0} are constant in the atmosphere.

We will assume that the latter two terms stay away from zero in the atmosphere ($r \geq r_a$). To make this assumption more explicit, we write out their expression here, cf. (D.20) and (D.23)

$$\begin{aligned}
C_{22} &= -\frac{\sigma^2}{\mathrm{c}_0^2} - \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\Phi_0'}{\mathrm{c}_0^2}\frac{1}{r}\,;\\
r^2\, C_{22}\, \hat{q}(r) &= -r^2\, C_{22} \,+\, \ell(\ell+1) \,=\, \frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \frac{\alpha_{\rho_0}}{\gamma}r \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2}r\,;\\
r^2\, C_{22}\, q(r) &= C_{22}(\alpha_{\rho_0}\, r^2 \,-\, 2\, r) \,+\, r\, C_{12} \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} - \frac{\alpha_{\rho_0}}{\gamma}\Big)\,;\\
r^2\, C_{22}\, \tilde{q}(r) &= r^2 C_{22} C_{11} \,+\, \ell(\ell+1)\left[\Big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\Big)\Big(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\Big) - \frac{2}{r^2}\right] + \Big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\Big) r C_{12}\,.
\end{aligned} \tag{7.56}$$

7.3.1 Discussion on the existence of singularity

When $\Gamma \neq 0$, we have

$$C_{22} \neq 0 \quad , \quad r^2 C_{22}\hat{q} \neq 0\,, \qquad \forall\, r \geq r_a\,. \tag{7.57}$$

Thus, in this case, the ODE (7.6) on the interval $[r_a, \infty)$ has no singularity.

When $\Gamma = 0$, $\sigma = \omega$, and we have the following observations.

1. We consider the equation $r^2 C_{22}\hat{q} = 0$. We first note that since we are in the atmosphere $r > 0$, thus

$$\begin{aligned} r^2\, C_{22}(r)\, \hat{q}(r) \;=\; 0 \quad &\Leftrightarrow \quad \frac{\omega^2}{\mathrm{c}_0^2} r \left(r \;+\; \frac{\alpha_{\rho_0}}{\gamma}\frac{\mathrm{c}_0^2}{\omega^2} \;-\; \frac{\Phi_0'}{\omega^2} \right) \;=\; 0 \\ &\Leftrightarrow \quad r \;+\; \frac{\alpha_{\rho_0}}{\gamma}\frac{\mathrm{c}_0^2}{\omega^2} \;-\; \frac{\Phi_0'}{\omega^2} \;=\; 0 \quad , \quad r > 0\,. \end{aligned} \tag{7.58}$$

   Since $r \mapsto \Phi_0'(r)$ is monotone (in fact strictly decreasing, see Figure 6), the equation

$$r \;-\; \frac{\Phi_0'(r)}{\omega^2} \;=\; -\frac{\alpha_{\rho_0}}{\gamma}\frac{\mathrm{c}_0^2}{\omega^2} \tag{7.59}$$

   either has no root or has exactly one root on $r \geq r_a$. If it exists, we denote by $r^\star_{\mathrm{a1},\omega}$ the unique simple root of (7.59), i.e.

$$\boxed{r^\star_{\mathrm{a1},\omega} \;=\; -\frac{\alpha_{\rho_0}}{\gamma}\frac{\mathrm{c}_0^2}{\omega^2} \;+\; \frac{\Phi_0'(r^\star_{\mathrm{a1},\omega})}{\omega^2}\,.} \tag{7.60}$$

   The existence of this zero is further discussed below using the solar parameter values of model `S+AtmoCAI`. Additionally, the zeros of $r^2 C_{22}\hat{q} = 0$ are the same for all ℓ, since the expression is independent of ℓ. This means that dividing by $r^2 C_{22}\hat{q}$ introduces at most a simple pole. It is also convenient to rewrite (7.60) as

$$\frac{\omega^2}{\mathrm{c}_0^2} r^\star_{\mathrm{a1},\omega} \;=\; -\,\frac{\alpha_{\rho_0}}{\gamma} \;+\; \frac{\Phi_0'(r^\star_{\mathrm{a1},\omega})}{\mathrm{c}_0^2}\,. \tag{7.61}$$

2. For $\ell = 0$, although the zero of $C_{22}(r) = 0$ coincides with that of (7.59), we do not have to worry about this creating more singularity since $r^2 C_{22} q$ and $r^2 C_{22}\tilde{q}$ do not have C_{22}'/C_{22} in their expression, in particular,

$$r^2\, C_{22}\, q(r) \;=\; C_{22}(\alpha_{\rho_0}\, r^2 \;-\; 2\,r) \;+\; r\, C_{12}\,, \qquad \ell = 0\,; \tag{7.62a}$$

$$r^2\, C_{22}\, \tilde{q}(r) \;=\; r^2 C_{22} C_{11}\,, \qquad \ell = 0\,. \tag{7.62b}$$

   In another word, the singularity of these terms only comes from $r^\star_{\mathrm{a1},\omega}$ at which it is a pole of order 1.

3. Consider the equation $C_{22}(r) = 0$ on $r \geq r_a$, which is equivalent to

$$\frac{\omega^2}{\mathrm{c}_0^2} r \;-\; \frac{\ell(\ell+1)}{r} \;-\; \frac{\Phi_0'(r)}{\mathrm{c}_0^2} \;=\; -\frac{\alpha_{\rho_0}}{\gamma}\,. \tag{7.63}$$

   Since $r \mapsto \frac{\omega^2}{\mathrm{c}_0^2} r \;+\; \frac{\alpha_{\rho_0}}{\gamma} r$ and the function $r \mapsto \frac{\Phi_0'(r)}{\mathrm{c}_0^2}$ is strictly decreasing, $r \mapsto -\frac{\Phi_0'(r)}{\mathrm{c}_0^2}$ is increasing. The following functions are also increasing

$$r \mapsto \frac{\omega^2}{\mathrm{c}_0^2} r \quad \text{, and} \quad r \mapsto -\,\frac{\ell(\ell+1)}{r}\,. \tag{7.64}$$

   Thus the left-hand-side of (7.63) is an increasing function. This means that (7.63) has at most one zero. Denoting by $r^\star_{\mathrm{a2},\omega,\ell}$ this unique simple zero on (r_a, ∞), if it exists, we have,

$$\boxed{\frac{\omega^2}{\mathrm{c}_0^2} r^\star_{\mathrm{a2},\omega,\ell} \;-\; \frac{\ell(\ell+1)}{r^\star_{\mathrm{a2},\omega,\ell}} \;-\; \frac{\Phi_0'(r^\star_{\mathrm{a2},\omega,\ell})}{\mathrm{c}_0^2} \;=\; -\frac{\alpha_{\rho_0}}{\gamma}\,.} \tag{7.65}$$

   The existence of this root is further discussed and illustrated in Subsection 7.5.1.

4. For $\ell > 0$, the zeros of $C_{22}(r) = 0$ cannot coincide with that of $r^2 C_{22}(r)\hat{q}(r) = 0$. This is seen by using the second expression of $r^2 C_{22}\hat{q}$ in (7.56),

$$r^2\, C_{22}(r)\, \hat{q}(r) \;=\; -r^2\, C_{22}(r) \;+\; \ell(\ell+1)\,. \tag{7.66}$$

Thus

$$(r^{\star}_{\mathrm{a1},\omega})^2\, C_{22}(r^{\star}_{\mathrm{a1},\omega})\, \hat{q}(r^{\star}_{\mathrm{a1},\omega}) \;=\; 0 \quad\Leftrightarrow\quad C_{22}(r^{\star}_{\mathrm{a1},\omega}) \;=\; \frac{\ell(\ell+1)}{(r^{\star}_{\mathrm{a1},\omega})^2} \tag{7.67}$$

and

$$C_{22}(r^{\star}_{\mathrm{a2},\omega,\ell}) \;=\; 0 \quad\Leftrightarrow\quad (r^{\star}_{\mathrm{a2},\omega,\ell})^2\, C_{22}(r^{\star}_{\mathrm{a2},\omega,\ell})\, \hat{q}(r^{\star}_{\mathrm{a2},\omega,\ell}) \;=\; \ell(\ell+1)\,. \tag{7.68}$$

In another word,

$$\boxed{r^{\star}_{\mathrm{a1},\omega} \neq r^{\star}_{\mathrm{a2},\omega,\ell}\,, \qquad \ell > 0\,.} \tag{7.69}$$

Existence of $r^{\star}_{\mathbf{a2},\omega,\ell}$ We consider the following assumption,

Assumption 4.

$$\frac{1}{4\,r_a} \;+\; \frac{\alpha_{\rho_0}}{\gamma} \;-\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2} \;>\; 0\,. \tag{7.70}$$

Under the above assumption, we define,

$$\boxed{\ell^{\star}_{a,\omega} \;:=\; \sqrt{\frac{\omega^2}{\mathrm{c}_0^2}(r_a)^2 \;+\; \frac{1}{4} \;+\; \left(\frac{\alpha_{\rho_0}}{\gamma} \;-\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2}\right) r_a} \;-\; \frac{1}{2}\,.} \tag{7.71}$$

Assumption 4 guarantees the positivity of the term in the square root.

Proposition 12. *Under Assumption 4 and $\Gamma = 0$, we have the following equivalence,*

$$\text{(7.63) has a unique zero on } (r_a, \infty) \qquad\Leftrightarrow\qquad \ell > \ell^{\star}_{a,\omega}\,. \tag{7.72}$$

Proof. **Statement** ($\Rightarrow$) Suppose (7.63) has a unique zero on (r_a, ∞). We have denoted this unique zero by $r^{\star}_{\mathrm{a2},\omega,\ell}$. We have

$$r^{\star}_{\mathrm{a2},\omega,\ell} \;=\; \left(\frac{\ell(\ell+1)}{r^{\star}_{\mathrm{a2},\omega,\ell}} \;+\; \frac{\Phi_0'(r^{\star}_{\mathrm{a2},\omega,\ell})}{\mathrm{c}_0^2} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2}\,. \tag{7.73}$$

Since $r^{\star}_{\mathrm{a2},\omega,\ell} \geq r_a$, so is the right-hand-side of the above equality, i.e.

$$\left(\frac{\ell(\ell+1)}{r^{\star}_{\mathrm{a2},\omega,\ell}} \;+\; \frac{\Phi_0'(r^{\star}_{\mathrm{a2},\omega,\ell})}{\mathrm{c}_0^2} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2} \;\geq\; r_a\,. \tag{7.74}$$

Since the right-hand-side is a decreasing function on (r_a, ∞), the above inequality occurs if

$$\left(\frac{\ell(\ell+1)}{r_a} \;+\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2} \;>\; \left(\frac{\ell(\ell+1)}{r^{\star}_{\mathrm{a2},\omega,\ell}} \;+\; \frac{\Phi_0'(r^{\star}_{\mathrm{a2},\omega,\ell})}{\mathrm{c}_0^2} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2} \;\geq\; r_a \tag{7.75}$$

which leads to

$$\left(\frac{\ell(\ell+1)}{r_a} \;+\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2} \;\geq\; r_a \tag{7.76a}$$

$$\Leftrightarrow\; \ell(\ell+1) \;\geq\; r_a\left(\frac{\omega^2}{\mathrm{c}_0^2}r_a \;-\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2} \;+\; \frac{\alpha_{\rho_0}}{\gamma}\right) \tag{7.76b}$$

$$\Leftrightarrow \ell \;+\; \frac{1}{2} \;\geq\; \sqrt{r_a\left(\frac{\omega^2}{\mathrm{c}_0^2}r_a \;-\; \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2} \;+\; \frac{\alpha_{\rho_0}}{\gamma} \;+\; \frac{1}{4\,r_a}\right)}\,. \tag{7.76c}$$

Statement ($\Leftarrow$) We now assume that $\ell > \ell^{\star}_{a,\omega}$. It suffices to prove the existence since uniqueness is discussed in the initial observations. We consider function

$$f \,:\, r \mapsto r \,-\, \left(\frac{\ell(\ell+1)}{r} \,+\, \frac{\Phi_0'(r)}{\mathrm{c}_0^2} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{\mathrm{c}_0^2}{\omega^2}$$

is continuous on $[r_a, \infty)$. If $\ell \geq \ell^{\star;a}_{\omega}$ then $f(r_a) < 0$. Since Φ_0' is of order r^{-2}, for large enough r, $f(r) > 0$. With f being continuous, $f(r) = 0$ thus has at least one zero on $[r_a, \infty)$.

□

Existence of $r^{\star}_{\mathbf{a1},\omega}$ The existence of the singularity depends on the choice of model of parameters. We first picture the evolution of Φ_0' in the atmosphere in Figure 6, using its expression in Lemma 1 and the model parameters given in (6.33).

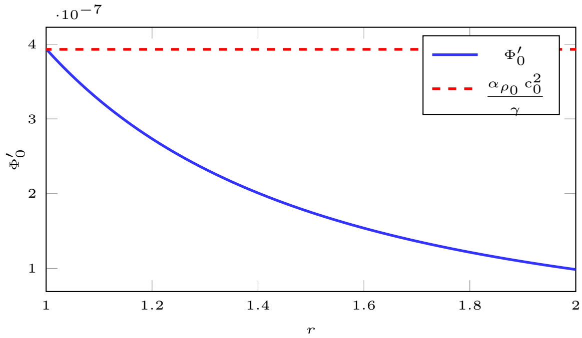


Figure 6: Evolution of Φ_0' (Lemma 1) in the solar atmosphere, using the model parameters given in (6.33).

We see that Φ_0' is a strictly decreasing function, and we define the following assumptions for the investigation of the existence of the singularity $r^{\star}_{\mathrm{a1},\omega}$.

Assumption 5.

$$\Phi_0'(r_a) \,\geq\, \frac{\alpha_{\rho_0}\,\mathrm{c}_0^2}{\gamma}\,. \tag{7.77}$$

or

Assumption 6.

$$\Phi_0'(r_a) \,<\, \frac{\alpha_{\rho_0}\,\mathrm{c}_0^2}{\gamma}\,. \tag{7.78}$$

Because Φ_0' is a strictly decreasing, there exists $r_{a1,\max}$ so that

$$\Phi_0'(r) \,-\, \frac{\alpha_{\rho_0}\,\mathrm{c}_0^2}{\gamma} \,<\, 0 \quad , \quad r \in (r_{a1,\max}, \infty)\,. \tag{7.79}$$

Under Assumption 5, $r_{a1,\max}$ is the unique value where

$$\Phi_0'(r_{a1,\max}) \,=\, \frac{\alpha_{\rho_0}\,\mathrm{c}_0^2}{\gamma}\,. \tag{7.80}$$

In the case of the solar atmospheric model `AtmoCAI` that we prescribed in (6.33), Assumption 5 is verified and we have:

$$\Phi_0'(r_a) \,=\, 3.931\,71 \times 10^{-7} \qquad \text{and} \qquad \frac{\alpha_{\rho_0}\,\mathrm{c}_0^2}{\gamma} = 3.930\,92 \times 10^{-7}\,. \tag{7.81}$$

Let us note that the difference between the two quantities is small (7×10^{-11}) and therefore, it is strongly related to the choice of model, and the representation with splines we have employed for the model `S` (see Appendix F). Clearly, this condition may not be validated by any stellar models. Then we obtain

$$r_{a1,\max} = 1.000\,812\,9\,, \qquad \text{for the model of (6.33).} \tag{7.82}$$

We see that the singularity, if it exists, is very near the beginning of the atmosphere, as we remind that $r_a = 1.000\,712\,6$. This is due to the small difference between the quantities of (7.81).

Under Assumption 5, we define

$$\omega_{a1}^{*} \;:=\; \sqrt{\frac{1}{r_a}\left(\Phi_0'(r_a) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\right)}\,. \tag{7.83}$$

This limiting frequency can be explicitly computed for our model `AtmoCAI` using the parameter values of (6.33) and we obtain,

$$\frac{\omega_{a1}^{*}}{2\,\pi} \;=\; 1.4126\ \mu\mathrm{Hz}\,, \qquad \text{limiting frequency for } r^{\star}_{a1,\omega} \text{ using model } \texttt{AtmoCAI}. \tag{7.84}$$

Therefore, in the case of our atmospheric model `AtmoCAI`, this singularity exists only at low frequency and, when it exists, it is near the beginning of the atmosphere.

Proposition 13. *In the following statements, we suppose* $\Gamma = 0$.

1. *We have the following equivalence*

$$\text{Equation (7.59) has a unique zero on } (r_a,\infty) \qquad \Leftrightarrow \qquad \text{Assumption 5 and } \omega \;<\; \omega_{a1}^{*}\,. \tag{7.85}$$

   Additionally, the zero of (7.59), denoted by $r^{\star}_{a1,\omega}$*, if it exists, is unique and has the further property that*

$$r^{\star}_{a1,\omega} \;\in\; (r_a,\, r_{a1,\max}]\,. \tag{7.86}$$

2. *This also means that, for the versions of model that satisfy Assumption 6, the equation (7.59) has no zero on* (r_a,∞).

Proof. **(Statement $\Rightarrow$)** If $r^{\star}_{a1,\omega}$ exists, then

$$r^{\star}_{a1,\omega} \;=\; \frac{1}{\omega^2}\left(\Phi_0'(r^{\star}_{a1,\omega}) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\right). \tag{7.87}$$

Since $r^{\star}_{a1,\omega} > r_a$, the right-hand-side is also greater than $> r_a$. In addition, since $r \mapsto \Phi_0'(r)$ in Figure 6 is a strictly decreasing function, we have

$$r_a \;<\; \frac{1}{\omega^2}\left(\Phi_0'(r^{\star}_{a1,\omega}) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\right) \;<\; \frac{1}{\omega^2}\left(\Phi_0'(r_a) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\right) \tag{7.88}$$

This leads immediately to condition

$$\omega^2 < \frac{1}{r_a}\left(\Phi_0'(r_a) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\right) \quad \text{, and} \quad \Phi_0'(r_a) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma} \;>\; 0\,. \tag{7.89}$$

This also gives us the statement regarding the interval to which $r^{\star}_{a1,\omega}$ belongs, which is

$$r^{\star}_{a1,\omega} \;\in\; \{ r > r_a \,|\, \Phi_0'(a) \;-\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma} \;>\; 0 \}. \tag{7.90}$$

Statement (7.86) follows from the strictly decreasing of Φ_0'.

(Statement $\Leftarrow$) We now assume Assumption 5 and $\omega \leq \omega_{a1}^{*}$. Consider the continuous function

$$f \;:\; r \;\mapsto\; r \;-\; \Phi_0'(r_a) \;+\; \frac{\alpha_{\rho_0}\,c_0^2}{\gamma}\,.$$

We have $f(r_a) < 0$. On the other hand, since $r \mapsto \Phi_0'(r)$ is of order r^{-2} as $r \to \infty$, $f(r) > 0$ for large enough r. Since f is continuous, this means $f(r) = 0$ has at least a zero on $[r_a, \infty)$. □

7.3.2 Computation of indicial exponents

In the following propositions, we consider the cases in which the equation (7.59) or (7.63) has a zero, and compute the corresponding indicial exponents.

Proposition 14. *In the case that equation (7.59) has a zero, with $r^{\star}_{a1,\omega}$ denoting the unique zero, cf. (7.60), we have*

$$\lim_{r\to r^{\star}_{a1,\omega}} (r - r^{\star}_{a1,\omega}) \frac{r^2\, C_{22}\, q}{r^2\, C_{22}\, \hat{q}} = 1 \tag{7.91}$$

and

$$\lim_{r\to r^{\star}_{a1,\omega}} (r - r^{\star}_{a1,\omega})^2 \frac{r^2\, C_{22}\, \tilde{q}}{r^2\, C_{22}\, \hat{q}} = 0\,. \tag{7.92}$$

Under this assumption, the ODE (7.6) on $r \geq r_a$, has a regular singularity at $r = r^{\star}_{a1,\omega}$ with indicial equation $\lambda^2 = 0$, and with double indicial exponent $\lambda = 0$.

Proof. From observation 1, we have $r^2 C_{22}\tilde{q}(r) = 0$ only has a pole of rank 1 at $r = r^{\star}_{\mathrm{a1},\omega}$, we thus have

$$\lim_{r\,\to\, r^{\star}_{\mathrm{a1},\omega}} \left(r - r^{\star}_{\mathrm{a1},\omega}\right)^2 \frac{r^2 C_{22}\, \tilde{q}}{r^2\, C_{22}\, \hat{q}} = 0\,.$$

It remains the consider the first statement.

Using l'Hopital's rule, we have

$$\begin{aligned}
\lim_{r\to r^{\star}_{\mathrm{a1},\omega}} (r - r^{\star}_{\mathrm{a1},\omega}) \frac{1}{r^2\, C_{22}\, \hat{q}} &= \lim_{r\to r^{\star}_{\mathrm{a1},\omega}} \frac{(r - r^{\star}_{\mathrm{a1},\omega})'}{(r^2\, C_{22}\, \hat{q})'} = \lim_{r\to r^{\star}_{\mathrm{a1},\omega}} \frac{1}{\frac{\omega^2}{\mathrm{c}_0^2} 2r + \frac{\alpha_{\rho_0}}{\gamma} - \frac{\Phi_0'(r)}{\mathrm{c}_0^2} - \frac{\Phi_0''(r)}{\mathrm{c}_0^2}\, r} \\
&= \frac{1}{\frac{\omega^2}{\mathrm{c}_0^2} r^{\star}_{\mathrm{a1},\omega} - \frac{\Phi_0''(r^{\star}_{\mathrm{a1},\omega})}{\mathrm{c}_0^2}\, r^{\star}_{\mathrm{a1},\omega}} = \frac{\mathrm{c}_0^2}{\left(\omega^2 - \Phi_0''(r^{\star}_{\mathrm{a1},\omega})\right) r^{\star}_{\mathrm{a1},\omega}} > 0\,.
\end{aligned} \tag{7.93}$$

In the third equality, we have used (7.61). The last inequality comes from the fact that $\Phi_0'' < 0$.

We next consider the limiting value of $r^2 C_{22} q$ at $r = r^{\star}_{\mathrm{a1},\omega}$. We substitute the definition of C_{12}, cf. e.g. (D.20), and using that $r^2 C_{22}(r) = \ell(\ell+1)$ at $r = r^{\star}_{\mathrm{a1},\omega}$, we have

$$\begin{aligned}
r^2\, C_{22}\, q(r)\Big|_{r=r^{\star}_{\mathrm{a1},\omega}} &= \ell(\ell+1)(\alpha_{\rho_0} - \frac{2}{r^{\star}_{\mathrm{a1},\omega}}) + r^{\star}_{\mathrm{a1},\omega}\, C_{12} + \ell(\ell+1)\Big(\frac{1}{r^{\star}_{\mathrm{a1},\omega}} - \frac{(r^{\star}_{\mathrm{a1},\omega})^2 C_{22}'(r^{\star}_{\mathrm{a1},\omega})}{\ell(\ell+1)} - \frac{\alpha_{\rho_0}}{\gamma}\Big) \\
&= \ell(\ell+1)(\alpha_{\rho_0} - \frac{2}{r^{\star}_{\mathrm{a1},\omega}}) + \ell(\ell+1)\left(-\alpha_{\rho_0} + \frac{\alpha_{\rho_0}}{\gamma} - \frac{1}{r^{\star}_{\mathrm{a1},\omega}}\right) \\
&\qquad + \ell(\ell+1)\Big(\frac{1}{r^{\star}_{\mathrm{a1},\omega}} - \frac{\alpha_{\rho_0}}{\gamma}\Big) - (r^{\star}_{\mathrm{a1},\omega})^2 C_{22}'(r^{\star}_{\mathrm{a1},\omega}) \\
&= -2\frac{\ell(\ell+1)}{r^{\star}_{\mathrm{a1},\omega}} - (r^{\star}_{\mathrm{a1},\omega})^2 C_{22}'(r^{\star}_{\mathrm{a1},\omega})\,.
\end{aligned} \tag{7.94}$$

From the expression of C_{22}' in (D.21), we have

$$\begin{aligned}
C_{22}'(r^{\star}_{\mathrm{a1},\omega}) &= \frac{1}{(r^{\star}_{\mathrm{a1},\omega})^2}\left(-2\frac{\alpha_{\rho_0}}{\gamma} + 3\left(\frac{\alpha_{\rho_0}}{\gamma} - \frac{\Phi_0'(r^{\star}_{\mathrm{a1},\omega})}{\mathrm{c}_0^2}\right)\right) + \frac{4\pi G \rho_0(r^{\star}_{\mathrm{a1},\omega})}{\mathrm{c}_0^2\, r^{\star}_{\mathrm{a1},\omega}} - 2\frac{\ell(\ell+1)}{(r^{\star}_{\mathrm{a1},\omega})^3} \\
&= \frac{1}{(r^{\star}_{\mathrm{a1},\omega})^2}\left(-2\frac{\alpha_{\rho_0}}{\gamma} - 3\frac{\omega^2}{\mathrm{c}_0^2} r^{\star}_{\mathrm{a1},\omega}\right) + 4\pi G\frac{\rho_0(r^{\star}_{\mathrm{a1},\omega})}{\mathrm{c}_0^2\, r^{\star}_{\mathrm{a1},\omega}} - 2\frac{\ell(\ell+1)}{(r^{\star}_{\mathrm{a1},\omega})^3}\,.
\end{aligned} \tag{7.95}$$

In the second equality, we have used (7.61). Using the value of C_{22}', we obtain

$$\begin{aligned}\Rightarrow r^2\, C_{22}\, q(r)\Big|_{r=r^\star_{\mathrm{a}1,\omega}} &= -2\frac{\ell(\ell+1)}{r^\star_{\mathrm{a}1,\omega}} + 2\frac{\alpha_{\rho_0}}{\gamma} + 3\frac{\omega^2}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega} - 4\pi G\frac{\rho_0(r^\star_{\mathrm{a}1,\omega})}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega} + 2\frac{\ell(\ell+1)}{r^\star_{\mathrm{a}1,\omega}} \\ &= 2\frac{\alpha_{\rho_0}}{\gamma} + 3\frac{\omega^2}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega} - \frac{\Phi_0''(r^\star_{\mathrm{a}1,\omega})}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega} - \frac{2}{\mathrm{c}_0^2}\Phi_0'(r^\star_{\mathrm{a}1,\omega}) \\ &= +\frac{\omega^2}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega} - \frac{\Phi_0''(r^\star_{\mathrm{a}1,\omega})}{\mathrm{c}_0^2} r^\star_{\mathrm{a}1,\omega}\,.\end{aligned} \tag{7.96}$$

In the second equation, we have replaced $4\pi G\rho_0 = \Phi_0'' + \frac{2}{r}\Phi_0'$ and used (7.61).

Putting together (7.93) and (7.96), we finally obtain

$$\lim_{r\to r^\star_{\mathrm{a}1,\omega}} (r - r^\star_{\mathrm{a}1,\omega})\,\frac{r^2\, C_{22}\, q}{r^2\, C_{22}\, \hat{q}} = \lim_{r\to r^\star_{\mathrm{a}1,\omega}} \frac{r - r^\star_{\mathrm{a}1,\omega}}{r^2\, C_{22}\, \hat{q}} \lim_{r\to r^\star_{\mathrm{a}1,\omega}} r^2\, C_{22}\, q = 1\,. \tag{7.97}$$

The associated indicial equation

$$\lambda\,(\lambda - 1) + \left(\lim_{r\to r^\star_{\mathrm{a}1,\omega}} (r - r^\star_{\mathrm{a}1,\omega})\,\frac{r^2\, C_{22}\, q}{r^2\, C_{22}\, \hat{q}}\right)\lambda = 0\,, \tag{7.98}$$

simplifies to

$$\lambda^2 = 0 \tag{7.99}$$

with double indicial exponents given by $\lambda = 0$.

□

Proposition 15. *If equation (7.63) has a zero, and with $r^\star_{a2,\omega,\ell}$ of (7.65) representing this unique zero, we have*

$$\lim_{r\to r^\star_{a2,\omega,\ell}} (r - r^\star_{a2,\omega,\ell})\,\frac{r^2\, C_{22}\, q}{r^2\, C_{22}\, \hat{q}} = -1\,, \tag{7.100}$$

and

$$\lim_{r\to r^\star_{a2,\omega,\ell}} (r - r^\star_{a2,\omega,\ell})^2\,\frac{r^2\, C_{22}\, \tilde{q}}{r^2\, C_{22}\, \hat{q}} = 0\,. \tag{7.101}$$

In this case, ODE (7.6) on $r \geq r_a$ has a regular singularity at $r = r^\star_{a2,\omega,\ell}$ with indicial exponent $\lambda = 0$ and $\lambda = 2$.

Proof. From observation 3, we have $C_{22}(r) = 0$ only has a pole of rank 1 at $r = r^\star_{\mathrm{a}2,\omega,\ell}$, we thus have

$$\lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} (r - r^\star_{\mathrm{a}2,\omega,\ell})^2\,\frac{r^2\, C_{22}\, \tilde{q}}{r^2\, C_{22}\, \hat{q}} = 0\,.$$

It remains the consider the first statement.

Consider,

$$\lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} (r - r^\star_{\mathrm{a}2,\omega,\ell})\,\frac{r^2\, C_{22}\, q}{r^2\, C_{22}\, \hat{q}} = \left(\lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} (r - r^\star_{\mathrm{a}1,\omega})\, r^2\, C_{22}\, q\right)\left(\lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} r^2\, C_{22}\, \hat{q}\right). \tag{7.102}$$

We rearrange the first term on the right-hand-side of (7.102),

$$r^2\, C_{22}\, q(r) = \frac{C_{22}\left(C_{22}(\alpha_{\rho_0} r^2 - 2r) + r\, C_{12} + \ell(\ell+1)\left(\frac{1}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\right) - C_{22}'\ell(\ell+1)}{C_{22}}\,. \tag{7.103}$$

Thus

$$\begin{aligned}\lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} r^2\, C_{22}\, q(r) &= \lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} \frac{C_{22}'}{C_{22}}\ell(\ell+1)\,; \\ \lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} (r - r^\star_{\mathrm{a}2,\omega,\ell}) r^2\, C_{22}\, q(r) &= \lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} \frac{r - r^\star_{\mathrm{a}2,\omega,\ell}}{C_{22}} \times \lim_{r\to r^\star_{\mathrm{a}2,\omega,\ell}} (-C_{22}')\ell(\ell+1) = -\ell(\ell+1)\,.\end{aligned} \tag{7.104}$$

On the other hand, using its definition in (7.65),

$$\lim_{r\to r^\star_{\mathrm{a2},\omega,\ell}} r^2\, C_{22}\, \hat{q} \,=\, \ell(\ell+1)\,, \tag{7.105}$$

and

$$\lim_{r\to r^\star_{\mathrm{a2},\omega,\ell}} (r\,-\,r^\star_{\mathrm{a2},\omega,\ell})\,\frac{r^2\,C_{22}\,q}{r^2\,C_{22}\,\hat{q}} \,=\, -1\,. \tag{7.106}$$

The indicial root associated to this point is

$$\lambda\,(\lambda\,-\,1)\,+\,\lim_{r\to r^\star_{\mathrm{a1},\omega}} (r\,-\,r^\star_{\mathrm{a2},\omega,\ell})\,\frac{r^2\,C_{22}\,q}{r^2\,C_{22}\,\hat{q}}\,\lambda \,=\, 0\,, \tag{7.107}$$

with indicial exponents

$$\lambda = 0\,, \qquad \lambda = 2\,. \tag{7.108}$$

□

7.4 Indicial analysis for the conjugate equation

The indicial analysis for the radial modal equation (7.5) transfers readily to that for the conjugate one (7.2), particular for V given by (8.3)

$$V_\ell(r) \,=\, \frac{1}{4}\mathfrak{h}_\ell^2(r)\,-\,\frac{1}{2}\partial_r\,\mathfrak{h}_\ell(r)\,+\,\mathfrak{g}_\ell(r)\,. \tag{7.109}$$

The function $\mathfrak{h}(r) = -\frac{q(r)}{\hat{q}(r)}$ has a simple pole at $r = 0$ and is smooth elsewhere, and $\mathfrak{g}(r) = -\frac{\tilde{q}(r)}{\hat{q}(r)}$. Thus then V_ℓ has a pole of order two at $r = 0$. The same reasoning applies to the other singular points.

It remains to calculate the indicial exponents, which are now zeros of

$$s(s\,-\,1)\,+\,\lim_{r\to 0} r^2\,V_\ell(r) \,=\, 0\,. \tag{7.110}$$

The indicial exponents associated to

$$r^\star \in \{r^\star_{\mathrm{i},\omega,\ell}\,,\,r^\star_{\mathrm{a2},\omega,\ell}\,,\,r^\star_{\mathrm{a1},\omega}\}\,, \tag{7.111}$$

are the zeros of

$$s(s\,-\,1)\,+\,\lim_{r\to 0}\,(r\,-\,r^\star)^2\,V_\ell(r) \,=\, 0\,. \tag{7.112}$$

We denote the roots of (7.110) and (7.112) respectively as

$$\tilde{\lambda}_0^\pm \qquad \text{and} \qquad \tilde{\lambda}_\star^\pm\,, \tag{7.113}$$

with the convention

$$\mathrm{Re}\,\tilde{\lambda}_\bullet^- \,<\, \mathrm{Re}\,\tilde{\lambda}_\bullet^+\,. \tag{7.114}$$

Recall that from Propositions 10 and 11, that

$$\eta_0 \,=\, \lim_{r\to 0} r\,\mathfrak{h}\,, \qquad \tilde{\eta}_0 \,=\, \lim_{r\to 0} r^2\,\mathfrak{g}\,, \tag{7.115}$$

and from Propositions 10, 11, 14 and 15,

$$\eta \,=\, \lim_{r\to 0}(r\,-\,r^\star_{\mathrm{i},\omega,\ell})\,\mathfrak{h}\,, \qquad 0 \,=\, \lim_{r\to 0}(r\,-\,r^\star_{\mathrm{i},\omega,\ell})^2\,\mathfrak{g}\,. \tag{7.116}$$

We write $\mathfrak{h}$ as

$$\mathfrak{h}(r) \,=\, \frac{\eta_0}{r}\,+\,\tilde{\mathfrak{h}}(r)\,, \qquad \tilde{\mathfrak{h}} \text{ regular at } r = 0\,. \tag{7.117}$$

Then

$$\frac{d}{dr}\mathfrak{h} \,=\, -\frac{\eta_0}{r^2}\,+\,\frac{d}{dr}\tilde{\mathfrak{h}}\,, \tag{7.118}$$

and

$$\lim_{r\to 0} r^2 \,\frac{d}{dr}\mathfrak{h} \;=\; \lim_{r\to 0} r\,\mathfrak{h} \;=\; \eta_0\,. \tag{7.119}$$

Similarly at $r^\star$.

$$\lim_{r\to r^\star} (r\,-\,r^\star)^2 \,\frac{d}{dr}\mathfrak{h} \;=\; \lim_{r\to r^\star} (r\,-\,r^\star)\,\mathfrak{h} \;=\; \eta_\star\,. \tag{7.120}$$

The indicial equation (7.110) takes explicit form,

$$\boxed{s(s\,-\,1) \,+\, \frac{1}{4}\eta_0^2 \,-\, \frac{1}{2}\eta_0 \,+\, \tilde{\eta}_0 \;=\; 0\,,} \tag{7.121}$$

and for $r = r_\star$,

$$\boxed{s(s\,-\,1) \,+\, \frac{1}{4}\eta_\star^2 \,-\, \frac{1}{2}\eta_\star \;=\; 0\,.} \tag{7.122}$$

We can also calculate the indicial exponents $\tilde{\lambda}_0^\pm$ and $\tilde{\lambda}_\star^\pm$ of the conjugate ODE, starting from those of the original ODE. We only need to keep track of what the Liouville factor, $e^{-\frac{1}{2}\int^r \mathfrak{h}(s)\,ds}$ contributes. Using the form (7.117) of $\mathfrak{h}$,

$$\int^r \mathfrak{h}(s)\,ds \;=\; \int^r \big(\tfrac{\eta_0}{s} \,+\, \tilde{\mathfrak{h}}(s)\big)\,ds \;=\; \eta_0 \log x \,+\, \int^r \tilde{\mathfrak{h}}(s)\,ds \tag{7.123}$$

$$\Rightarrow \quad e^{-\frac{1}{2}\int^r \mathfrak{h}(s)\,ds} \;=\; e^{-\frac{1}{2}\eta_0 \log x}\, e^{-\frac{1}{2}\int^r \tilde{\mathfrak{h}}(s)\,ds} \;=\; x^{-\frac{1}{2}\eta_0} \times \begin{pmatrix}\text{a regular function}\\ \text{not vanishing at } 0\end{pmatrix}\,. \tag{7.124}$$

We have similar results in a small neighborhood of $r = r^\star$. Thus the indicial exponents of the conjugate ODE are obtained from those of the original ODE by relation,

$$\boxed{\tilde{\lambda}_0^\pm \;=\; \lambda_0^\pm \,-\, \frac{1}{2}\eta_0\,, \qquad \lambda_\star^\pm \;=\; \lambda_\star^\pm \,-\, \frac{1}{2}\eta\,.} \tag{7.125}$$

In particular, for $r = 0$, since $\eta_0 = 2$ for $\ell = 0$ and 4 for $\ell > 0$, the first relation is

$$\tilde{\lambda}_0^\pm \;=\; \lambda_0^\pm \,-\, 1\,,\ \text{for } \ell = 0 \quad ; \quad \tilde{\lambda}_0^\pm \;=\; \lambda_0^\pm \,-\, 2\,,\ \text{for } \ell > 0\,, \tag{7.126}$$

and

$$\tilde{\lambda}_0^- \;=\; -3\,, \qquad \tilde{\lambda}_0^+ \;=\; 0\,. \tag{7.127}$$

We summarize the above discussion in the following proposition.

Proposition 16. *The poles of V_ℓ are given as follows:*

1. *For $\Gamma > 0$ (i.e., with attenuation), V_ℓ only has a pole at $r = 0$ on $(0,\infty)$, and this is a pole of order 2.*
2. *For $\Gamma = 0$, in addition to $r = 0$, V can have pole of order 2 at the following positions,*

$$0 \,<\, r^\star_{\mathrm{i},\omega,\ell} \,<\, r_a\,, \qquad r_a \,<\, r^\star_{a1,\omega} \quad \textit{and} \quad r_a \,<\, r^\star_{a2,\omega,\ell}\,. \tag{7.128}$$

This means that the potential V_ℓ is continuous and bounded on $[r_{reg},\infty)$ with

$$r_{reg} \quad \textit{arbitrarily small} \qquad \textit{for } \Gamma > 0, \tag{7.129}$$

and

$$r_{reg} \,>\, \max\{r^\star_{\mathrm{i},\omega,\ell}\,,\, r^\star_{a1,\omega}\,,\, r^\star_{a2,\omega,\ell}\} \qquad \textit{for } \Gamma = 0\,, \tag{7.130}$$

if these zeros exist.

7.5 Numerical illustrations with the solar model

Following our analysis of the singularity of the potential V_ℓ and Proposition 16, we provide some numerical illustrations, where we use the background parameters of the Sun, that are shown in Figures 1 to 3, see also Appendix F. We first investigate the behavior of the singularities, before plotting the potential. Let us first recall that for the singularity $r^\star_{\mathrm{a1},\omega}$, it only exists at low frequency, (7.84), and, when it exists, it remains near the beginning of the atmosphere.

7.5.1 Singularities $r^\star_{\mathrm{i},\omega,\ell}$ and $r^\star_{\mathbf{a2},\omega,\ell}$

We consider the case without attenuation. Let us first recall the definition of $\ell^\star_\omega$, which is the maximal mode at which $r^\star_{\mathrm{i},\omega,\ell}$ exists, cf. (7.20), with $\ell^\star_{a,\omega}$, the minimal mode at which $r^\star_{\mathrm{a2},\omega,\ell}$ exists, given by (7.71):

$$\ell^\star_\omega = -\frac{1}{2} + \sqrt{\frac{\omega^2\, r_a^2}{\mathrm{c}_0(r_a)^2} + \frac{1}{4}}\,, \qquad \ell^\star_{a,\omega} = -\frac{1}{2} + \sqrt{\frac{\omega^2\, r_a^2}{\mathrm{c}_0(r_a)^2} + \frac{1}{4} + \left(\frac{\alpha_{\rho_0}}{\gamma} - \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2(r_a)}\right) r_a}\,.$$

Comparing $\ell^\star_\omega$ with $\ell^\star_{a,\omega}$, we observe that

$$\text{Assumption 5} \quad \Longrightarrow \quad \ell^\star_\omega \geq \ell^\star_{a,\omega}\,, \tag{7.131a}$$

$$\text{Assumption 6} \quad \Longrightarrow \quad \ell^\star_\omega < \ell^\star_{a,\omega}\,. \tag{7.131b}$$

We have seen that our model `AtmoCAI` verifies Assumption 5 and injecting the values from (7.81), we have

$$r_a \left(\frac{\alpha_{\rho_0}}{\gamma} - \frac{\Phi_0'(r_a)}{\mathrm{c}_0^2(r_a)}\right) = -0.81\,. \tag{7.132}$$

In fact, because the modes are integer, we observe a continuity in the singularity between $r^\star_{\mathrm{i},\omega,\ell}$ and $r^\star_{\mathrm{a2},\omega,\ell}$, such that $\ell^\star_\omega = \ell^\star_{a,\omega}$. When $\ell \leq \ell^\star_\omega$, the singularity in the interior, $r^\star_{\mathrm{i},\omega,\ell}$, and $r^\star_{\mathrm{a2},\omega,\ell}$ does not exist. On the other hand, when $\ell > \ell^\star_\omega$, the singularity is moved towards the atmosphere, in $r^\star_{\mathrm{a2},\omega,\ell}$ while $r^\star_{\mathrm{i},\omega,\ell}$ does not exist.

In Figure 7, we picture the position of the singularity $r^\star_{\mathrm{i},\omega,\ell}$ or $r^\star_{\mathrm{a2},\omega,\ell}$, for modes between $\ell = 1$ and $\ell = 2000$, and for frequencies from 0.1 mHz to 12 mHz. We use the values of the parameters from the solar model `S` for the interior, and from model `AtmoCAI` for the atmosphere.

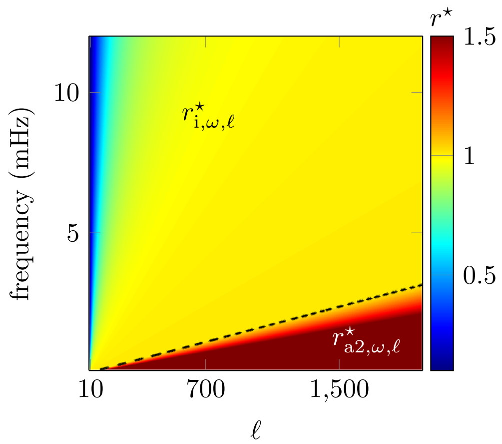
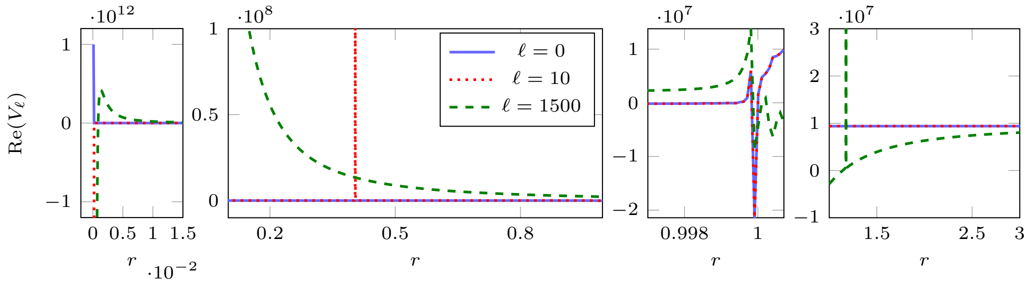


Figure 7: Position of the singularity $r^\star_{\mathrm{i},\omega,\ell}$ or $r^\star_{\mathrm{a2},\omega,\ell}$ for the solar model. We investigate modes from $\ell = 1$ to $\ell = 2000$ and frequencies from 0.1 mHz to 12 mHz. The black dashed line indicates the separation between the singularity located in the interior (above the line: $r^\star_{\mathrm{i},\omega,\ell}$) or in the atmosphere (below the line: $r^\star_{\mathrm{a2},\omega,\ell}$), and corresponds to $\ell^\star_\omega$ of (7.20). For visualization, all the positions where $r^\star_{\mathrm{a2},\omega,\ell} > 1.5$ uses the same color, while the maximum, obtained at $\ell = 2000$ for frequency 0.1 mHz is $r^\star_{\mathrm{a2},0.1\mathrm{mHz},\ell=2000} = 30.89$.

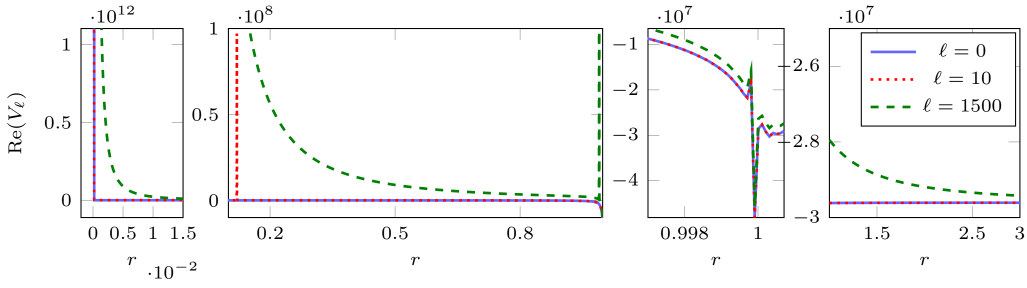
We see that the singularity is mainly positioned near the $r = 1$. It is moved towards the origin when the frequency increases and when the mode decreases. Here, the minimum is obtained at frequency 12 mHz for mode $\ell = 1$ with $r^\star_{\mathrm{i},12\mathrm{mHz},\ell=1} = 1.36 \times 10^{-2}$. On the other hand, it is moved away with increasing mode and decreasing frequency, the maximum is obtained for 0.1 mHz for mode $\ell = 2000$ with $r^{\star}_{\mathrm{a2,0.1mHz},\ell=2000} = 30.89$.

7.5.2 Evaluation of the potential

Eventually, we provide the evaluations of the real part of the potential in the interior and atmosphere, where we consider r from 0 to 3. In Figure 8, we picture the potential without attenuation ($\Gamma = 0$) at frequencies 2 and 10 mHz for different modes. The case with attenuation, using $\Gamma/(2\pi) = 20$ µHz is pictured in Figure 9. For visualization (i.e., to allow different scales), our figures are separated into four panels: one corresponds to near the origin, one for the interior, one near the interface with the atmosphere (i.e. near r_a) and one for the atmosphere. We use frequencies in mHz, which are typical of applications in helioseismology, therefore, we do not observe the singularity in $r^{\star}_{\mathrm{a1},\omega}$ which requires much lower frequencies (lower than 1.4 µHz as given in (7.84)).



(a) $\omega/(2\pi) = 2$ mHz, $\Gamma = 0$.

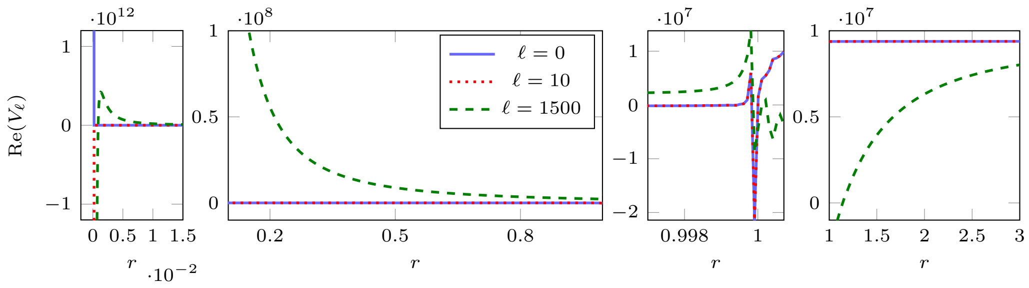


(b) $\omega/(2\pi) = 10$ mHz, $\Gamma = 0$.

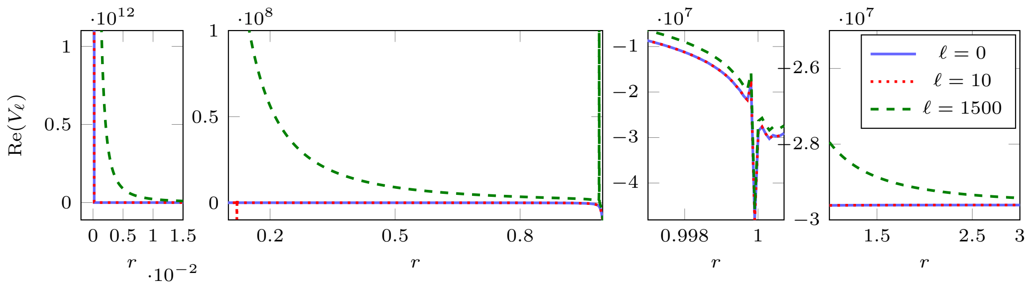
Figure 8: Solar potential V_ℓ depending on the mode ℓ and the frequency, assuming there is no attenuation, such that $\Gamma = 0$. For visualization, the interval is split in four for the different regions. In addition to the singularity in $r = 0$, the singularity in $r = r^{\star}_{\mathrm{i},\omega,\ell}$ or $r = r^{\star}_{\mathrm{a2},\omega,\ell}$ is shown by the peaks, and depends on the mode and the frequency.

For all choices of frequency and mode, we observe the singularity at the origin (in $r = 0$). In the case without attenuation, Figure 8, we see the additional singularity in $r^{\star}_{\mathrm{i},\omega,\ell}$ or $r^{\star}_{\mathrm{a2},\omega,\ell}$: this singularity moves towards the exterior when the mode increases, or when the frequency decreases. Namely, for mode $\ell = 10$, the singularity is in the interior (that is, $r^{\star}_{\mathrm{i},\omega,\ell}$) for the two frequencies (2 mHz and 10 mHz), and appears closer to the origin at 10 mHz. For higher mode $\ell = 1500$, the singularity is in the atmosphere (that is, $r^{\star}_{\mathrm{a2},\omega,\ell}$) at frequency 2 mHz and in the interior (that is, $r^{\star}_{\mathrm{i},\omega,\ell}$) at 10 mHz. These observations coincide with the numerical evaluation of the singularity provided in Figure 7. Furthermore, we note that only the singularity at the origin remains at mode $\ell = 0$ without attenuation (Figure 8) or when we incorporate attenuation, see Figure 9. We note that, near the surface and due to the variation of the physical properties, the potential shows some important changes in all cases.

In the case with attenuation, we see some variations at the position where a singularity is in the case without attenuation. This is due to the relatively small imaginary part added by the attenuation. Nonetheless, these are *not* singularity, and they have finite values. We illustrate in Figure 10 where we zoom near the singularity for mode $\ell = 10$ at 10 mHz. We see that without attenuation, we have a sharp peak that increases towards infinity, while it is a Gaussian-shape function with attenuation.

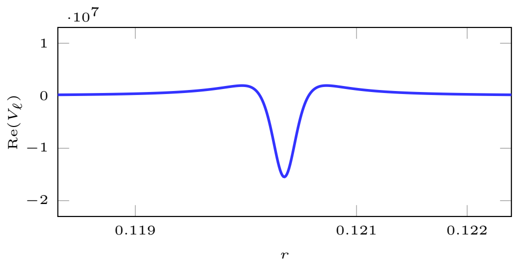


(a) $\omega/(2\pi)$ =2 mHz, $\Gamma/(2\pi)$ = 20 µHz.

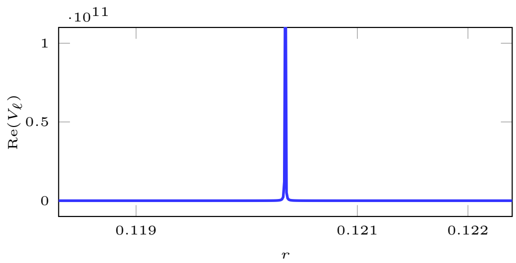


(b) $\omega/(2\pi)$ =10 mHz, $\Gamma/(2\pi)$ = 20 µHz.

Figure 9: Solar potential V_ℓ for the interior depending on the mode ℓ and the frequency, using attenuation $\Gamma/(2\pi)$ = 20 µHz. For visualization, the interval is split in four for the different regions.



(a) Using $\Gamma/(2\pi)$ = 20 µHz.



(b) Using $\Gamma = 0$.

Figure 10: Zoom near $r^\star_{\mathrm{i},\omega,\ell}$ of the Solar potential V_ℓ in the interior at frequency 10 mHz and mode $\ell = 10$.

8 Analysis of the modal ODE: asymptotic

In this section, we obtain explicitly the first three terms in the asymptotic expansion of V_ℓ in (7.2) as $r \to \infty$, such that,

$$V_\ell(r) \,=\, \mathsf{v}_0 \,+\, \frac{\mathsf{v}_{-1}}{r} \,+\, \frac{\mathsf{v}_{-2}}{r^2} \,+\, \mathsf{O}(r^{-3})\,, \qquad r \to \infty\,. \tag{8.1}$$

Let us first recall that the conjugated unknown $\tilde{a}(r)$ solves the conjugate ODE

$$-\partial_r^2 \tilde{a} \,+\, V_\ell(r)\,\tilde{a} \,=\, 0\,, \tag{8.2}$$

with

$$V_\ell(r) \,=\, \frac{1}{4}\mathfrak{h}^2(r) \,-\, \frac{1}{2}\partial_r \mathfrak{h}(r) \,+\, \mathfrak{g}(r)\,. \tag{8.3}$$

We will present two approaches. In the first one, the asymptotic analysis is done without using explicit expression for $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{g}$ obtained Proposition 8, by keeping track of only top order terms. This gives an approximation result with error of order r^{-3}, cf. Proposition 19. On the other hand, taking advantage of the special assumptions of `AtmoCAI`, one can also carry out approximation directly on the expressions given Proposition 8. The second approach, available under this specific assumption, allows for higher order approximation and explicit description of the error. It also allows us to present an enriched approximation of order r^{-3}, which makes appear the effects of gravity, cf. (8.65) or (8.66). We note that the usual $\mathsf{O}(r^{-3})$ as obtained in Approach 1 works well at infinity, but does not show the effect of gravity which comes into presence at order r^{-3} and only for $\mathfrak{g}$.

We also recall the notation introduced in (4.50),

$$k_0 := \frac{\sigma}{\mathrm{c}_0} \tag{8.4}$$

8.1 Approach 1

8.1.1 Asymptotic of C_{22}

To begin with, we write out the asymptotic of coefficient C_{22}.

Proposition 17. *For all $k_0 \neq 0$,*

$$\frac{1}{C_{22}} = -\frac{1}{k_0^2}\left(1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + \left(\ell(\ell+1) + \frac{\alpha_{\rho_0}^2}{\gamma^2\,(k_0)^2}\right)\frac{1}{(k_0 r)^2} + k_0^{-2}\mathsf{O}(r^{-3})\right), \qquad r\to\infty\,. \tag{8.5}$$

Here the error $\mathsf{O}(r^{-3})$ is bounded independently of k_0. For $k_0 > \hat{k}_0$,

$$\frac{1}{C_{22}} = \frac{1}{k_0^2}\,\mathsf{O}(1)\,, \qquad r\to\infty\,. \tag{8.6}$$

On the other hand, C_{22}' is independent of k_0, and

$$C_{22}' = \pi\, G\,\frac{\rho_0}{\mathsf{c}_0^2}\,\frac{1}{r} + \mathsf{O}(r^{-2})\,, \tag{8.7}$$

thus

$$\left(\frac{1}{r\, C_{22}}\right)' = k_0^{-2}\,\mathsf{O}(r^{-2})\,. \tag{8.8}$$

Proof. Denote by $\varepsilon(r)$ the following function independent of k_0:

$$\varepsilon(r) := \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} - \frac{\ell(\ell+1)}{r^2} - \frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r}\,. \tag{8.9}$$

From Lemma 1, we have

$$\frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r} = \mathsf{O}(r^{-3})\,. \tag{8.10}$$

For fixed ℓ, this means

$$\varepsilon = r^{-1}\,\mathsf{O}(1) = \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} + \mathsf{O}(r^{-2}) = \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} - \frac{\ell(\ell+1)}{r^2} + \mathsf{O}(r^{-3})\,, \qquad r\to\infty\,, \tag{8.11}$$

and

$$\varepsilon^2 = \frac{\alpha_{\rho_0}^2}{\gamma^2}\frac{1}{r^2} + \mathsf{O}(r^{-3})\,, \qquad \varepsilon^3 = \mathsf{O}(r^{-3})\,, \qquad r\to\infty\,. \tag{8.12}$$

As a result of this, for

$$C_{22} = -k_0^2 - \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r} = -k_0^2 - \varepsilon(r)\,, \tag{8.13}$$

we have

$$\begin{aligned} -k_0^2\frac{1}{C_{22}} &= \frac{1}{1 + k_0^{-2}\varepsilon(r)} = 1 - k_0^{-2}\varepsilon(r) + k_0^{-4}\varepsilon^2(r) + k_0^{-6}\mathsf{O}(|\varepsilon(r)|^3) \\ &= 1 - k_0^{-2}\Big(\frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} - \frac{\ell(\ell+1)}{r^2} + \mathsf{O}(r^{-3})\Big) + k_0^{-4}\Big(\frac{\alpha_{\rho_0}^2}{\gamma^2}\frac{1}{r^2} + \mathsf{O}(r^{-3})\Big) + k_0^{-6}\mathsf{O}(r^{-3})\,. \end{aligned} \tag{8.14}$$

For $k_0 > \hat{k}_0$, we have

$$\begin{aligned}\frac{1}{C_{22}} &= -\frac{1}{k_0^2}\left(1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + \frac{\ell(\ell+1)}{(k_0 r)^2} + \frac{\alpha_{\rho_0}^2}{\gamma^2\,(k_0)^2}\frac{1}{(k_0 r)^2} + k_0^{-2}\mathsf{O}(r^{-3})\right)\\ &= -\frac{1}{k_0^2}\left(1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + \left(\ell(\ell+1) + \frac{\alpha_{\rho_0}^2}{\gamma^2\,(k_0)^2}\right)\frac{1}{(k_0 r)^2} + k_0^{-2}\mathsf{O}(r^{-3})\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}C_{22}' &= \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2} - 2\frac{\ell(\ell+1)}{r^3} + \frac{1}{\mathsf{c}_0^2}\frac{\Phi_0''}{r} - \frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r^2}\\ &= \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2} - 2\frac{\ell(\ell+1)}{r^3} + \frac{1}{\mathsf{c}_0^2}\frac{1}{r}(4\pi G\rho_0 - \frac{2}{r}\Phi_0') - \frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r^2}\\ &= 4\pi G\frac{\rho_0}{\mathsf{c}_0^2}\frac{1}{r} + \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2} - 2\frac{\ell(\ell+1)}{r^3} - 3\frac{1}{\mathsf{c}_0^2}\frac{\Phi_0'}{r^2}.\end{aligned} \tag{8.15}$$

Thus C_{22}' is independent of k_0 and

$$C_{22}' = \pi G\frac{\rho_0}{\mathsf{c}_0^2}\frac{1}{r} + \mathsf{O}(r^{-2}). \tag{8.16}$$

Since

$$\left(\frac{1}{r\, C_{22}}\right)' = -\frac{1}{r^2}\frac{1}{C_{22}} - \frac{1}{C_{22}^2}\frac{C_{22}'}{r},$$

for $k_0 > \hat{k}_0$, this leads to,

$$\left(\frac{1}{r\, C_{22}}\right)' = \frac{1}{r^2}\frac{1}{k_0^2}\mathsf{O}(1) + \frac{1}{k_0^4}\mathsf{O}(1)\mathsf{O}(r^{-2}) = k_0^{-2}\mathsf{O}(r^{-2}).$$

□

8.1.2 Asymptotic of $\hat{q}$, $\tilde{q}$ and q

We write out the asysmptotics for the coefficients $\hat{q}$, $\tilde{q}$ and q.

Lemma 3. *The coefficients of the ODE (7.1) have the following asymptotic expansion, as $r \to \infty$,*

$$-\frac{1}{\hat{q}(r)} = 1 - \frac{\ell(\ell+1)}{(k_0 r)^2} + k_0^{-2}\,\mathsf{O}(r^{-3}), \tag{8.17a}$$

$$q(r) = \alpha_{\rho_0} - \frac{2}{r} + \frac{\ell(\ell+1)\,\alpha_{\rho_0}}{k_0^2}\frac{1}{r^2} + k_0^{-2}\,\mathsf{O}(r^{-3}), \tag{8.17b}$$

$$\tilde{q}(r) = -\frac{\sigma^2}{\mathsf{c}_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} + \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}^2\,(1-\gamma)}{\gamma^2}\frac{1}{r^2} + \mathsf{O}(r^{-3}) + k_0^{-2}\,\mathsf{O}(r^{-3}). \tag{8.17c}$$

Proof. Let us consider $-\frac{1}{\hat{q}(r)}$, from (8.5), we have

$$\frac{1}{r^2\, C_{22}} = -\frac{1}{k_0^2}\frac{1}{r^2}\left(1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + k_0^{-2}\mathsf{O}(r^{-2})\right), \tag{8.18}$$

and for $k_0 > \hat{k}_0$,

$$\frac{1}{r^2\, C_{22}} = -\frac{1}{k_0^2}\frac{1}{r^2}\,\mathsf{O}(1) \quad\Longrightarrow\quad \left(\frac{1}{r^2\, C_{22}}\right)^2 = \frac{1}{r^4}\frac{1}{k_0^4}\,\mathsf{O}(1). \tag{8.19}$$

As a result of (8.18), using Neumann series expansion, we obtain

$$-\frac{1}{\hat{q}(r)} = \frac{1}{1 - \dfrac{\ell(\ell+1)}{r^2\, C_{22}}} = 1 + \frac{\ell(\ell+1)}{r^2\, C_{22}} + \mathsf{O}\Big(\Big|\frac{\ell(\ell+1)}{r^2\, C_{22}}\Big|^2\Big) = 1 - \frac{\ell(\ell+1)}{(k_0 r)^2} + \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{\ell(\ell+1)}{(k_0\, r)^3} + k_0^{-4}\mathsf{O}(r^{-4}),$$

or simply

$$-\frac{1}{\hat{q}(r)} = 1 - \frac{\ell(\ell+1)}{(k_0\, r)^2} + k_0^{-4}\,\mathsf{O}(r^{-3}). \tag{8.20}$$

Then, since

$$\left(\frac{1}{r\, C_{22}}\right)' = k_0^{-2}\mathsf{O}(r^{-2}),$$

we have

$$q(r) = \alpha_{\rho_0} - \frac{2}{r} + \frac{1}{r}\frac{C_{12}}{C_{22}} - \ell(\ell+1)\frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2 C_{22}} + \underbrace{\frac{\ell(\ell+1)}{r}\left(\frac{1}{r\, C_{22}}\right)' + \frac{2\,\ell(\ell+1)}{r^3 C_{22}}}_{k_0^{-2}\,\mathsf{O}(r^{-3})}.$$

It remains to consider $\frac{C_{12}}{C_{22}}$. Using the definition of C_{12} in (4.39c),

$$C_{12} = \ell(\ell+1)\left(\frac{1}{r}\left(-\alpha_{\rho_0} + \frac{\alpha_{\rho_0}}{\gamma}\right) - \frac{1}{r^2}\right),$$

and (8.5) which gives the asymptotic of C_{22}^{-1},

$$\begin{aligned}
\frac{C_{12}}{C_{22}} &= C_{12}C_{22}^{-1} = \ell(\ell+1)\left(\frac{1}{r}\left(-\alpha_{\rho_0} + \frac{\alpha_{\rho_0}}{\gamma}\right) - \frac{1}{r^2}\right)\left(-\frac{1}{k_0^2}\right)\left(1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + k_0^{-2}\mathsf{O}(r^{-2})\right) \\
&= -\frac{\ell(\ell+1)}{k_0^2}\left[\frac{\alpha_{\rho_0}}{\gamma}\frac{(1-\gamma)}{r} - \frac{1}{r^2}\right]\left[1 - \frac{\alpha_{\rho_0}}{\gamma\, k_0}\frac{1}{k_0\, r} + k_0^{-2}\mathsf{O}(r^{-2})\right] \\
&= -\frac{\ell(\ell+1)}{k_0^2}\left(\frac{\alpha_{\rho_0}}{\gamma}\frac{(1-\gamma)}{r} - \left(\frac{\alpha_{\rho_0}^2(1-\gamma)}{\gamma^2\, k_0^2} + 1\right)\frac{1}{r^2} + \frac{\alpha_{\rho_0}}{\gamma\, k_0^{-2}}\frac{1}{r^3} + k_0^{-2}\,\mathsf{O}(r^{-4})\right) \\
&= -\frac{\ell(\ell+1)}{k_0^2}\left(\frac{\alpha_{\rho_0}}{\gamma}\frac{(1-\gamma)}{r} - \left(\frac{\alpha_{\rho_0}^2(1-\gamma)}{\gamma^2\, k_0^2} + 1\right)\frac{1}{r^2} + k_0^{-2}\,\mathsf{O}(r^{-3})\right).
\end{aligned}$$

This gives

$$\frac{1}{r}\frac{C_{12}}{C_{22}} = -\frac{\ell(\ell+1)}{k_0^2}\left(\frac{\alpha_{\rho_0}}{\gamma}\frac{(1-\gamma)}{r^2} - \left(\frac{\alpha_{\rho_0}^2(1-\gamma)}{\gamma^2\, k_0^2} + 1\right)\frac{1}{r^3} + k_0^{-2}\,\mathsf{O}(r^{-4})\right). \tag{8.21}$$

In summary, we have

$$q(r) = \alpha_{\rho_0} - \frac{2}{r} + \left(-\frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}\,(1-\gamma)}{\gamma} + \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}}{\gamma}\right)\frac{1}{r^2} + k_0^{-2}\,\mathsf{O}(r^{-3}),$$

which simplifies to

$$q(r) = \alpha_{\rho_0} - \frac{2}{r} + \frac{\ell(\ell+1)\,\alpha_{\rho_0}}{k_0^2}\frac{1}{r^2} + k_0^{-2}\,\mathsf{O}(r^{-3}). \tag{8.22}$$

Eventually, since

$$\frac{1}{r}\left[\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{1}{rC_{22}}\right]' = -\frac{1}{r}\frac{2}{r^2}\frac{1}{rC_{22}} + \left(\frac{2}{r} + \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{1}{r}\left(\frac{1}{rC_{22}}\right)' = k_0^{-2}\,\mathsf{O}(r^{-3}),$$

we have

$$\tilde{q}(r) = C_{11} + \underbrace{\frac{\ell(\ell+1)}{r}\left[\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{1}{rC_{22}}\right]' + \frac{2}{r}\frac{C_{12}}{rC_{22}}}_{\ell(\ell+1)\,k_0^{-2}\,\mathsf{O}(r^{-3})} - \frac{\alpha_{\rho_0}}{\gamma}\frac{C_{12}}{rC_{22}}.$$

It remains to look at C_{11}, given by

$$C_{11} = -\frac{\sigma^2}{\mathsf{c}_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} - \frac{2}{\mathsf{c}_0^2}\underbrace{\frac{\Phi_0'}{r}}_{\mathsf{O}(r^{-3})+\text{expo decay}} + \frac{4\pi G}{\mathsf{c}_0^2}\underbrace{\rho_0}_{\text{decays exponentially}}.$$

Thus

$$C_{11} \sim -\frac{\sigma^2}{\mathsf{c}_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} + \mathsf{O}(r^{-3}), \qquad r \to \infty. \tag{8.23}$$

Together with (8.21), we obtain

$$\tilde{q}(r) = -\frac{\sigma^2}{\mathsf{c}_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} + \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}^2\,(1-\gamma)}{\gamma^2}\frac{1}{r^2} + \mathsf{O}(r^{-3}) + k_0^{-2}\,\mathsf{O}(r^{-3}). \tag{8.24}$$

□

8.1.3 Asymptotic of the coefficients of the ODE

Here, we obtain the asymptotic for the coefficient of the normalized ODE (4.62),

$$-\partial_r^2 a + \mathfrak{h}(r)\,a + \mathfrak{g}(r) = 0, \tag{8.25}$$

with

$$\mathfrak{h}(r) := -\frac{q(r)}{\hat{q}(r)}, \qquad \mathfrak{g}(r) := -\frac{\tilde{q}(r)}{\hat{q}(r)}. \tag{8.26}$$

We label the coefficients of the asymptotic expansions at infinity of the ODE (7.1) in (8.17) as

$$\begin{aligned}
&\check{q}_{-2} := -\frac{\ell(\ell+1)}{k_0^2},\\
&q_0 = \alpha_{\rho_0}, \qquad q_{-1} = -2, \qquad q_{-2} := \frac{\alpha_{\rho_0}\,\ell(\ell+1)}{k_0^2} = -\alpha_{\rho_0}\,\check{q}_{-2},\\
&\tilde{q}_0 = -k_0^2, \qquad \tilde{q}_{-1} = 2\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right),\\
&\tilde{q}_{-2} = 2 + \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}^2\,(1-\gamma)}{\gamma^2} = 2 - \frac{\alpha_{\rho_0}^2\,(1-\gamma)}{\gamma^2}\check{q}_{-2}.
\end{aligned} \tag{8.27}$$

Proposition 18. *The coefficient of ODE (8.25) has the following asymptotic as $r \to \infty$,*

$$\begin{aligned}
\mathfrak{h}(r) &= \mathfrak{h}_0 + \frac{\mathfrak{h}_{-1}}{r} + \frac{\mathfrak{h}_{-2}}{r^2} + \mathsf{O}(r^{-3}),\\
\mathfrak{g}(r) &= \mathfrak{g}_0 + \frac{\mathfrak{g}_{-1}}{r} + \frac{\mathfrak{g}_{-2}}{r^2} + \mathsf{O}(r^{-3}),
\end{aligned} \tag{8.28}$$

with

$$\begin{aligned}
&\mathfrak{h}_0 = \alpha_{\rho_0}, \qquad \mathfrak{h}_{-1} = -2, \qquad \mathfrak{h}_{-2} = 0;\\
&\mathfrak{g}_0 = -k_0^2, \qquad \mathfrak{g}_{-1} = 2\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right), \qquad \mathfrak{g}_{-2} = 2 + \ell(\ell+1) + \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}^2(1-\gamma)}{\gamma^2}.
\end{aligned} \tag{8.29}$$

Proof. We have

$$\begin{aligned}
\mathfrak{h}(r) &= \left(1 + \frac{\check{q}_{-2}}{r^2} + k_0^{-2}\,\mathsf{O}(r^{-3})\right)\left(q_0 + \frac{q_{-1}}{r} + \frac{q_{-2}}{r^2} + k_0^{-2}\mathsf{O}(r^{-3})\right)\\
\Rightarrow \mathfrak{h}(r) &= q_0 + \frac{q_{-1}}{r} + r^{-2}\big(q_0\,\check{q}_{-2} + q_{-2}\big) + k_0^{-2}\,\mathsf{O}(r^{-3}).
\end{aligned}$$

Note that $\check{q}_{-2}$ contains a factor of k_0^{-2} which allows for the presence of this factor in the error.

From (8.27), we have the simplification

$$\check{q}_{-2}\, q_0 + q_{-2} \;=\; \check{q}_{-2}\, \alpha_{\rho_0} \;-\; \alpha_{\rho_0} \check{q}_{-2} \;=\; 0\,.$$

Similarly, we have

$$\mathfrak{g}(r) \;=\; \Big(1 \,+\, \frac{\check{q}_{-2}}{r^2} \,+\, k_0^{-2}\, \mathsf{O}(r^{-3})\Big) \Big(\tilde{q}_0 + \frac{\tilde{q}_{-1}}{r} + \frac{\tilde{q}_{-2}}{r^2} \,+\, k_0^{-2}\mathsf{O}(r^{-3}) \,+\, \mathsf{O}(r^{-3})\Big)\,, \tag{8.30}$$

and thus,

$$\mathfrak{g} \;=\; \tilde{q}_0 \;+\; \frac{\tilde{q}_{-1}}{r} \;+\; \frac{\tilde{q}_0\, \check{q}_{-2} \,+\, \tilde{q}_{-2}}{r^2} \;+\; \mathsf{O}(r^{-3}) \;+\; k_0^{-2}\mathsf{O}(r^{-3})\,. \tag{8.31}$$

Simplification of the higher order terms gives,

$$\begin{aligned}
\tilde{q}_0\, \check{q}_{-2} \,+\, \tilde{q}_{-2} \;&=\; (-k_0^2)\, \check{q}_{-2} \,+\, 2 \,-\, \frac{\alpha_{\rho_0}^2\,(1-\gamma)}{\gamma^2}\, \check{q}_{-2} \;=\; -\check{q}_{-2} \left(k_0^2 \,+\, \frac{\alpha_{\rho_0}^2 (1-\gamma)}{\gamma^2}\right) \,+\, 2 \\
&=\; \frac{\ell(\ell+1)}{k_0^2} \left(k_0^2 \,+\, \frac{\alpha_{\rho_0}^2 (1-\gamma)}{\gamma^2}\right) \,+\, 2 \;=\; 2 \,+\, \ell(\ell+1) \,+\, \frac{\ell(\ell+1)}{k_0^2}\, \frac{\alpha_{\rho_0}^2 (1-\gamma)}{\gamma^2}\,.
\end{aligned}$$

□

8.1.4 Asymptotic of the potential

Combining all of the above asymptotic results, we obtain the one for the potential.

Proposition 19. *$V_\ell(r)$ defined in (8.3) has the following asymptotics as $r \to \infty$,*

$$V_\ell(r) \;=\; \mathsf{v}_0 + \frac{\mathsf{v}_{-1}}{r} \;+\; \frac{\mathsf{v}_{-2}}{r^2} \;+\; \mathsf{O}(r^{-3})\,, \qquad r \to \infty\,, \tag{8.32}$$

where

$$\begin{aligned}
\mathsf{v}_0 \;&=\; \frac{\alpha_{\rho_0}^2}{4} \,-\, \frac{\sigma^2}{\mathsf{c}_0^2} \;=\; \frac{\alpha_{\rho_0}^2}{4} \,-\, k_0^2\,, \\
\mathsf{v}_{-1} \;&=\; \alpha_{\rho_0} \,-\, 2\frac{\alpha_{\rho_0}}{\gamma}\,, \\
\mathsf{v}_{-2} \;&=\; 2 \,+\, \ell(\ell+1) \,+\, \frac{\ell(\ell+1)}{k_0^2}\, \frac{\alpha_{\rho_0}^2 (1-\gamma)}{\gamma^2}\,.
\end{aligned} \tag{8.33}$$

The error is bounded in k_0, for $k_0 > \hat{k}_0$.

Proof. To obtain the asymptotic expansion for V_ℓ as $r \to \infty$, we first recall from (8.29), that

$$\begin{aligned}
&\mathfrak{h}_0 \;=\; \alpha_{\rho_0}\,, \qquad \mathfrak{h}_{-1} = -2\,, \qquad \mathfrak{h}_{-2} = 0\,; \\
&\mathfrak{g}_0 \;=\; -k_0^2 \quad, \quad \mathfrak{g}_{-1} = 2\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right)\,, \qquad \mathfrak{g}_{-2} = 2 \,+\, \ell(\ell+1) \,+\, \frac{\ell(\ell+1)}{k_0^2}\, \frac{\alpha_{\rho_0}^2 (1-\gamma)}{\gamma^2}\,.
\end{aligned}$$

We thus obtain the asymptotic for $(\mathfrak{h}(r))^2$ as

$$(\mathfrak{h}(r))^2 \;=\; \mathfrak{h}_0^2 + \frac{2\,\mathfrak{h}_0\,\mathfrak{h}_{-1}}{r} \;+\; \frac{2\,\mathfrak{h}_0\,\mathfrak{h}_{-2} \,+\, (\mathfrak{h}_{-1})^2}{r^2} \;+\; \mathsf{O}(r^{-3})\,, \tag{8.34}$$

and that for $\partial_r \mathfrak{h}$,

$$\partial_r \mathfrak{h}(r) \;=\; \frac{\partial_r q(r)}{\hat{q}(r)} \;-\; \frac{q(r)\,\partial_r \hat{q}}{\hat{q}(r)^2} \qquad \Rightarrow \qquad \partial_r \mathfrak{h} = -q_{-1} r^{-2} + \mathsf{O}(r^{-3}) = -\mathfrak{h}_{-1} r^{-2} + \mathsf{O}(r^{-3})\,.$$

Putting these asymptotics together, we obtain that for V_ℓ:

$$\mathsf{v}_0 \;=\; \frac{1}{4}\mathfrak{h}_0^2 \;+\; \mathfrak{g}_0 \;=\; \frac{\alpha_{\rho_0}^2}{4} \;-\; k_0^2\,,$$

and

$$\mathsf{v}_{-1} \;=\; \frac{1}{4}\,(2\,\mathfrak{h}_0\,\mathfrak{h}_{-1}) \,+\, \mathfrak{g}_{-1} \;=\; -\alpha_{\rho_0} \,+\, 2\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) = \alpha_{\rho_0} \,-\, 2\frac{\alpha_{\rho_0}}{\gamma}\,.$$

On the other hand,

$$\begin{aligned} \upsilon_{-2} \;&=\; \frac{1}{4}\left(2\,\mathfrak{h}_0\,\mathfrak{h}_{-2} \,+\, (\mathfrak{h}_{-1})^2\right) \,-\, \frac{1}{2}(-\mathfrak{h}_{-1}) \,+\, \mathfrak{g}_{-2} \\ &=\; \frac{1}{4}\,(\mathfrak{h}_{-1})^2 \,-\, \frac{1}{2}(-\mathfrak{h}_{-1}) \,+\, \mathfrak{g}_{-2} \\ &=\; \mathfrak{g}_{-2}\,. \end{aligned}$$

□

8.2 Approach 2

We obtain asymptotic approximations for $\mathfrak{h}$, $\mathfrak{h}'$ and $\mathfrak{g}$, starting directly from their explicit expression given in Proposition 8. For simplicity, we work under the assumption of constant attenuation i.e.

$$(k_0^2)' \;=\; 0\,. \tag{8.35}$$

We first note that for $\ell = 0$,

$$\mathfrak{h}_0 \;=\; \alpha_{\rho_0} \;-\; \frac{2}{r}\,; \tag{8.36a}$$

$$\mathfrak{g}_0 \;=\; -k_0^2 \;+\; \frac{2\,(\alpha_{\rho_0} \,-\, \frac{\alpha_{\rho_0}}{\gamma})}{r} \;+\; \frac{2}{r^2}\,. \tag{8.36b}$$

It suffices to consider for higher ℓ.

We recall from Lemma 1 that in $r \geq r_a$,

$$\Phi_0' \;=\; \frac{G}{r^2}\mathfrak{m} \;+\; \text{exponential decay term} \tag{8.37}$$

thus

$$\Phi_0'' \;=\; -2\frac{G}{r^3}\mathfrak{m} \;+\; \text{exponential decay term}\,. \tag{8.38}$$

By its definition

$$E_{\text{he}} \;=\; -\frac{\alpha_{\rho_0}}{\gamma} \;+\; \frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\,\frac{1}{r^2} \;+\; \text{exponential decay term}\,. \tag{8.39}$$

Approximations with exponentially small error For rational approximations with exponentially small error, in expressions (4.76) (4.75) and (4.77), we replace

$$\Phi_0' \;\sim\; \frac{G}{r^2}\mathfrak{m} \quad ; \quad \Phi_0'' \;\sim\; -2\frac{G}{r^3}\mathfrak{m} \quad ; \quad E_{\text{he}} \;\sim\; -\frac{\alpha_{\rho_0}}{\gamma} \;+\; \frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\,\frac{1}{r^2}\,. \tag{8.40}$$

We denote these as

$$\boxed{\mathfrak{h}_{\text{app-exp}} \quad , \quad (\mathfrak{h}')_{\text{app-exp}} \quad , \quad \mathfrak{g}_{\text{app-exp}}} \tag{8.41}$$

and

$$\boxed{V_{\text{app-exp}} \;:=\; \frac{1}{4}(\mathfrak{h}_{\text{app-exp}})^2 \;-\; \frac{1}{2}(\mathfrak{h}')_{\text{app-exp}} \;+\; \mathfrak{g}_{\text{app-exp}}}\,. \tag{8.42}$$

Approximation with error $\mathsf{O}(r^{-3})$ In (4.75), the third term is of order r^{-3}, as $r \to \infty$. In addition, it is clear that they are bounded in k_0^{-2} for all $k_0 \geq 0$, thus we arrive at the same result from Proposition 18,

$$\boxed{\mathfrak{h} \;=\; \alpha_{\rho_0} \;-\; \frac{2}{r} \;+\; \frac{\ell(\ell+1)}{k_0^2}\,\mathcal{E}_{\mathfrak{h}}} \tag{8.43}$$

with error,

$$\mathcal{E}_{\mathfrak{h}} \;=\; \frac{2\,r \;-\; \frac{E_{\mathrm{he}}}{k_0^2} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\frac{1}{k_0^2}}{\left(r^2 \;-\; \frac{\ell(\ell+1)}{k_0^2} \;-\; \frac{r}{k_0^2}E_{\mathrm{he}}\right)\,\left(r^2 \;-\; \frac{r}{k_0^2}\,E_{\mathrm{he}}\right)} \;=\; \mathsf{O}(r^{-3}) \tag{8.44}$$
$$\text{bounded uniformly in } k_0^2 \in [0,\infty) \text{ and } \ell(\ell+1) \text{ for } \ell = 0,1,\ldots$$

Similarly,

$$\mathfrak{h}' \;=\; \frac{2}{r^2} \;+\; \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-4})\,. \tag{8.45}$$

with $\mathsf{O}(r^{-4})$ term bounded uniformly in $k_0^2 \in [0,\infty)$ and $\ell(\ell+1)$ for $\ell = 0,\ldots$. We write

$$\mathfrak{h}_{-3}^{\mathrm{app}} \;=\; \mathfrak{h}_0 \;=\; \alpha_{\rho_0} \;-\; \frac{2}{r}\,; \tag{8.46a}$$
$$(\mathfrak{h}')_{-3}^{\mathrm{app}} \;=\; \mathfrak{h}_0' \;=\; \frac{2}{r^2}\,. \tag{8.46b}$$

Since the term (4.76b) in expression (4.76) for $\mathfrak{g}$ is of order r^{-3}, we reobtain the result from Proposition 18. We introduce the notation

$$\boxed{\begin{aligned} \mathfrak{g}_{-3}^{\mathrm{app}} \;&=\; -k_0^2 \;+\; \frac{2\,(\alpha_{\rho_0} \;-\; \frac{\alpha_{\rho_0}}{\gamma})}{r} \;+\; \frac{2}{r^2} \;+\; \ell(\ell+1)\frac{k_0^2 \;+\; \frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \;-\; \alpha_{\rho_0}\right)}{k_0^2\,r^2 \;-\; r\,E_{\mathrm{he}}}\,; &&(8.47\mathrm{a})\\ \tilde{\mathfrak{g}}_{-3}^{\mathrm{app}} \;&=\; -k_0^2 \;+\; \frac{2\,(\alpha_{\rho_0} \;-\; \frac{\alpha_{\rho_0}}{\gamma})}{r} \;+\; \frac{1}{r^2}\left(2 \;+\; \ell(\ell+1) \;+\; \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \;-\; \alpha_{\rho_0}\right)\right) &&(8.47\mathrm{b})\end{aligned}}$$

then

$$\mathfrak{g} \;= \tilde{\mathfrak{g}}_{-3}^{\mathrm{app}} \;+\; \frac{\ell(\ell+1)}{k_0^2}\tilde{\mathcal{E}}_{\mathfrak{g},-3} \;=\; \mathfrak{g}_{-3}^{\mathrm{app}} \;+\; \frac{\ell(\ell+1)}{k_0^2}\mathcal{E}_{\mathfrak{g},-3} \tag{8.48}$$

where

$$\begin{aligned}\mathcal{E}_{\mathfrak{g}} \;&:=\; -\;\frac{\Phi_0''}{\mathrm{c}_0^2}\,\frac{1}{r^2 \;-\; \frac{r}{k_0^2}\,E_{\mathrm{he}}} \;+\; \frac{\Phi_0''}{\mathrm{c}_0^2}\,\frac{1}{k_0^2} \\ &\quad+\; \left(\frac{2}{r} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\,\frac{2r \;-\; k_0^{-2}\,E_{\mathrm{he}} \;-\; \frac{r}{k_0^2}\frac{\Phi_0''}{\mathrm{c}_0^2}}{\left(r^2 \;-\; \frac{\ell(\ell+1)}{k_0^2} \;-\; \frac{r}{k_0^2}E_{\mathrm{he}}\right)\,\left(r^2 \;-\; \frac{r}{k_0^2}\,E_{\mathrm{he}}\right)}\,.\end{aligned} \tag{8.49}$$

This error term also satisfies

$$\mathcal{E}_{\mathfrak{g}} \;=\; \mathsf{O}(r^{-3}) \text{ bounded uniformly in } k_0^2 \in [0,\infty) \text{ and } \ell(\ell+1) \text{ for } \ell = 0,1,\ldots \tag{8.50}$$

For V, we define

$$V_{-3}^{\mathrm{app}} \;=\; \frac{1}{4}(\mathfrak{h}_{-3}^{\mathrm{app}})^2 \;-\; \frac{1}{2}(\mathfrak{h}')_{-3}^{\mathrm{app}} \;+\; \mathfrak{g}_{-3}^{\mathrm{app}}\,; \tag{8.51a}$$
$$\tilde{V}_{-3}^{\mathrm{app}} \;=\; \frac{1}{4}(\mathfrak{h}_{-3}^{\mathrm{app}})^2 \;-\; \frac{1}{2}(\mathfrak{h}')_{-3}^{\mathrm{app}} \;+\; \mathfrak{g}_{-3}^{\mathrm{app}}\,. \tag{8.51b}$$

Since

$$\frac{1}{4}(\mathfrak{h}_{-3}^{\mathrm{app}})^2 \;-\; \frac{1}{2}(\mathfrak{h}')_{-3}^{\mathrm{app}} \;=\; \frac{\alpha_{\rho_0}^2}{4} \;-\; \frac{\alpha_{\rho_0}}{r}\,, \tag{8.52}$$

we have

$$V^{\mathrm{app}}_{-3} \;=\; \frac{\alpha_{\rho_0}^2}{4} \,-\, k_0^2 \,+\, \frac{\alpha_{\rho_0} \,-\, 2\frac{\alpha_{\rho_0}}{\gamma}}{r} \,+\, \frac{2}{r^2} \,+\, \ell(\ell+1)\frac{k_0^2 \,+\, \frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \,-\, \alpha_{\rho_0}\right)}{k_0^2\, r^2 \,-\, r\, E_{\mathrm{he}}}\,; \tag{8.53a}$$

$$\tilde{V}^{\mathrm{app}}_{-3} \;=\; \frac{\alpha_{\rho_0}^2}{4} \,-\, k_0^2 \,+\, \frac{\alpha_{\rho_0} \,-\, 2\frac{\alpha_{\rho_0}}{\gamma}}{r} \,+\, \frac{1}{r^2}\left(2 \,+\, \ell(\ell+1) \,+\, \frac{\ell(\ell+1)}{k_0^2}\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \,-\, \alpha_{\rho_0}\right)\right). \tag{8.53b}$$

We also have the same result as in Proposition 19,

$$V(r) \;=\; \tilde{V}^{\mathrm{app}}_{-3} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-3}) \;=\; V^{\mathrm{app}}_{-3} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-3})\,. \tag{8.54}$$

Higher order approximations By ignoring $\frac{\Phi_0''}{\mathrm{c}_0^2}r$ in the numerator of the last expression of $\mathfrak{h}$, we obtain an approximation of order r^{-6}. We further ignore the lower order term in E_{he} and obtain an approximation of order r^{-5}. Define

$$\mathfrak{h}^{\mathrm{app}}_{-6} \;:=\; \alpha_{\rho_0} \,-\, \frac{2}{r} \,+\, \ell(\ell+1)\frac{2\,k_0^2\, r \,-\, E_{\mathrm{he}}}{\big(k_0^2 r^2 \,-\, \ell(\ell+1) \,-\, rE_{\mathrm{he}}\big)\,\big(k_0^2\, r^2 \,-\, r\, E_{\mathrm{he}}\big)}\,, \tag{8.55}$$

and

$$\mathfrak{h}^{\mathrm{app}}_{-5} \;:=\; \alpha_{\rho_0} \,-\, \frac{2}{r} \,+\, \ell(\ell+1)\frac{2\,k_0^2\, r \,+\, \frac{\alpha_{\rho_0}}{\gamma}}{\big(k_0^2 r^2 \,-\, \ell(\ell+1) \,+\, r\frac{\alpha_{\rho_0}}{\gamma}\big)\,\big(k_0^2\, r^2 \,+\, r\,\frac{\alpha_{\rho_0}}{\gamma}\big)}\,. \tag{8.56}$$

We have

$$\mathfrak{h} \;=\; \mathfrak{h}^{\mathrm{app}}_{-6} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-6}) \;=\; \mathfrak{h}^{\mathrm{app}}_{-5} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-5})\,. \tag{8.57}$$

We have

$$\mathfrak{h}' \;=\; (\mathfrak{h}')^{\mathrm{app}}_{-5} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-5}) \tag{8.58}$$

with

$$(\mathfrak{h}')^{\mathrm{app}}_{-5} \;=\; \frac{2}{r^2} \,-\, \frac{6\,\ell(\ell+1)\,k_0^2}{\big(k_0^2 r^2 \,-\, \ell(\ell+1) \,+\, r\,\frac{\alpha_{\rho_0}}{\gamma}\big)\,\big(k_0^2\, r^2 \,+\, r\,\frac{\alpha_{\rho_0}}{\gamma}\big)}\,. \tag{8.59}$$

We did as above to obtain an approximation of order r^{-5} for $\mathfrak{g}$,

$$\begin{aligned}\mathfrak{g}^{\mathrm{app}}_{-5} \;:=\; &-\, k_0^2 \,+\, \frac{2\,(\alpha_{\rho_0} \,-\, \frac{\alpha_{\rho_0}}{\gamma})}{r} \,+\, \frac{2}{r^2} \,+\, \ell(\ell+1)\frac{k_0^2 \,+\, \frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \,-\, \alpha_{\rho_0}\right)}{k_0^2\, r^2 \,+\, r\,\frac{\alpha_{\rho_0}}{\gamma}} \\ &-2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3} \,+\, \ell(\ell+1)\left(\frac{2}{r} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{2\,k_0^2\, r \,+\, \frac{\alpha_{\rho_0}}{\gamma}}{\big(k_0^2 r^2 \,-\, \ell(\ell+1) \,+\, r\frac{\alpha_{\rho_0}}{\gamma}\big)\,\big(k_0^2\, r^2 \,+ r\,\frac{\alpha_{\rho_0}}{\gamma}\big)}\end{aligned} \tag{8.60}$$

then

$$\mathfrak{g} \;=\; \mathfrak{g}^{\mathrm{app}}_{-5} \,+\, \frac{\ell(\ell+1)}{k_0^2}\mathsf{O}(r^{-5})\,. \tag{8.61}$$

Potential V is then approximated by,

$$V^{\mathrm{app}}_{-5} \;=\; \frac{1}{4}(\mathfrak{h}^{\mathrm{app}}_{-5})^2 \,-\, \frac{1}{2}(\mathfrak{h}^{\mathrm{app}}_{-5})' \,+\, \mathfrak{g}^{\mathrm{app}}_{-5}\,. \tag{8.62}$$

Locally enriched $O(r^{-3})$ approximation Near $r = r_a$, with constant $2\frac{G\mathfrak{m}}{\mathrm{c}_0^2}$ being large, the term $-2\frac{G\mathfrak{m}}{\mathrm{c}_0}r^{-3}$ has significant effect in this region, which prevents $\mathfrak{g}_{-3}^{\mathrm{app}}$ from approximating $\mathfrak{g}$ well. The same poor approximation carries over for V. For $\mathfrak{h}$, since terms carrying $\frac{G\mathfrak{m}}{\mathrm{c}_0^2}$ appear at order r^{-5}, this influence is much more tamed. Because of this, $\mathfrak{h}_{-3}^{\mathrm{app}}$ approximates $\mathfrak{h}$ well even close to $r = 1$, despite the area around the spike when there are zeros of the denominator (i.e. at $r = r^{\star}_{a1,\omega,\ell}$ and $r = r^{\star}_{a2,\omega,\ell}$. Thus an enriched version near $r = 1$ with the same order of error at infinity (i.e. r^{-3}), we can work with

$$\mathfrak{h}_0 \ , \ \mathfrak{h}_0' \tag{8.63}$$

and

$$\mathfrak{g}_{G,-3}^{\mathrm{app}} \ := \ \mathfrak{g}_{-3}^{\mathrm{app}} \ - \ 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3} \qquad \text{or} \qquad \tilde{\mathfrak{g}}_{G,-3}^{\mathrm{app}} \ := \ \tilde{\mathfrak{g}}_{-3}^{\mathrm{app}} \ - \ 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3}\,. \tag{8.64}$$

The approximation for V are given by

$$\boxed{\tilde{V}_{G,-3}^{\mathrm{app}} \ = \ \tilde{V}_{-3}^{\mathrm{app}} \ - \ 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3}\,.} \tag{8.65}$$

or

$$\boxed{V_{G,-3}^{\mathrm{app}} \ = \ V_{-3}^{\mathrm{app}} \ - \ 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3}\,.} \tag{8.66}$$

8.3 Numerical illustration of the performance

We evaluate the quality of the asymptotics for the functions $\mathfrak{h}$, $\mathfrak{h}'$, $\mathfrak{g}$ and V_ℓ, using the expansions obtained at order 3 and 5, given above. In addition to the visualization of the functions, we also introduce the relative error between the function and its asymptotic, that depends on the frequency and mode, and that we define by

$$\mathfrak{e}_{V_\bullet}(r) = \frac{\big|V_\ell(r) - V_{\bullet,\ell}(r)\big|}{|V_\ell(r)|}\,, \tag{8.67}$$

and similarly with $\mathfrak{e}_{\mathfrak{h}_\bullet}$, $\mathfrak{e}_{\mathfrak{h}'_\bullet}$ and $\mathfrak{e}_{\mathfrak{g}_\bullet}$.

8.3.1 Numerical illustration for the asymptotic of $\mathfrak{h}$

We investigate the accuracy of the asymptotic expansion of $\mathfrak{h}$, given at order 3 and 5 respectively by (8.46a) and (8.56). In Figures 11 and 12, we picture the function $\mathfrak{h}$ together with its asymptotic $\mathfrak{h}_{-3}^{\mathrm{app}}$ and $\mathfrak{h}_{-5}^{\mathrm{app}}$, as well as their corresponding relative error $\mathfrak{e}_{\mathfrak{h}_\bullet}$, respectively without and with attenuation.

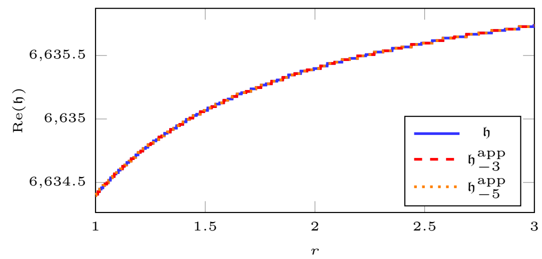
We observe that the asymptotic is equally accurate with or without attenuation and that, in both cases, it becomes less accurate at high modes. Namely, it reaches at best a relative error of 10^{-12} at mode $\ell = 0$, and 10^{-8} at mode $\ell =$1500. As expected, the asymptotic at order 5, $\mathfrak{h}_{-5}^{\mathrm{app}}$, gives the best results and we observe a gain of 2 orders of magnitude in terms of accuracy when we compare from order 3 and 5. The maximal error occurs near the singularity $r^{\star}_{\mathrm{a}2,\omega,\ell}$, if it exists (that is, only for high modes, as illustrated in Subsection 7.5).

We can conclude of the expansion of $\mathfrak{h}$ that the asymptotics accurately capture the behaviours. For small modes, the order 3 approximation, $\mathfrak{h}_{-3}^{\mathrm{app}}$, appears sufficient (about 10^{-8} relative error), but the order 5 is necessary for larger modes, as illustrated with $\ell = 1500$. Also, we do not observe any differences if attenuation is incorporated of not.

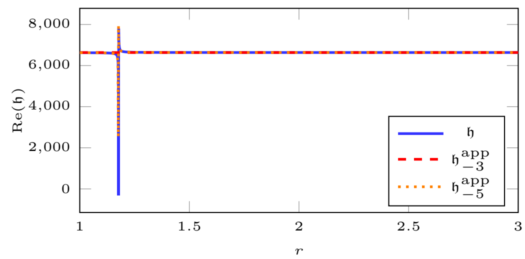
8.3.2 Numerical illustration for the asymptotic of $\mathfrak{h}'$

We now experiment with the asymptotic expansion of $\mathfrak{h}'$, given at order 3 and 5 respectively by (8.46b) and (8.59). In Figures 13 and 14, we picture the function together with its asymptotic $\mathfrak{h}'^{\mathrm{app}}_{-3}$ and $\mathfrak{h}'^{\mathrm{app}}_{-5}$, and the corresponding relative error $\mathfrak{e}_{\mathfrak{h}'_\bullet}$. The two figures correspond respectively to the case without and with attenuation.

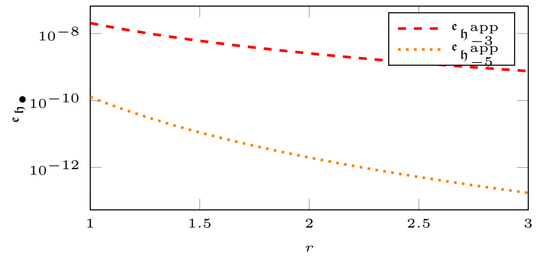
Similarly as for the function $\mathfrak{h}$, we do not see a difference in case with or without attenuation. On the other hand, we see that the relative error for the asymptotic expansion of $\mathfrak{h}'$ is larger than that for $\mathfrak{h}$, in particular for the high modes. Here, for the mode $\ell = 10$, the relative error reaches 10^{-8} while it is



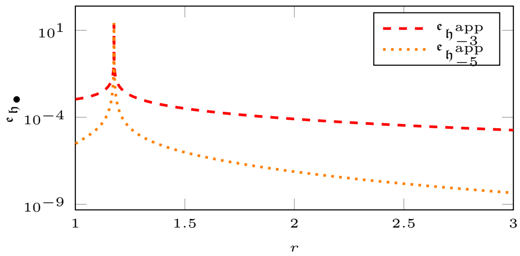
(a) Using frequency 2 mHz and mode $\ell = 10$.



(b) Using frequency 2 mHz and mode $\ell = 1500$.

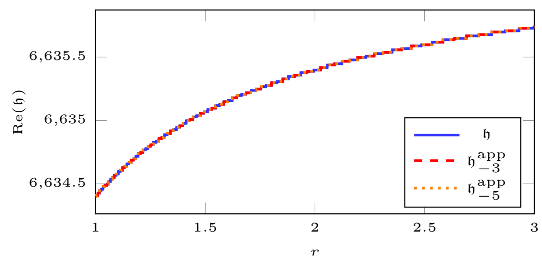


(c) Using frequency 2 mHz and mode $\ell = 10$.

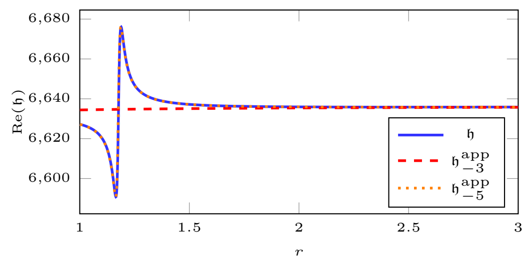


(d) Using frequency 2 mHz and mode $\ell = 1500$.

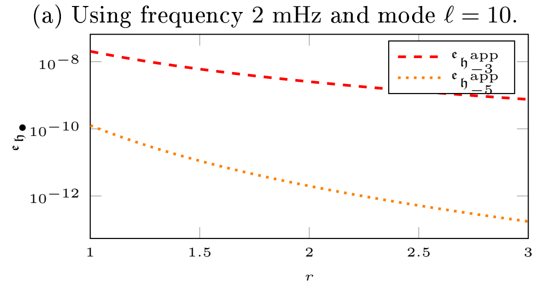
Figure 11: Evaluation of the asymptotic of $\mathfrak{h}$ at order 3 and 5, given by (8.46a) and (8.56) in the case without attenuation ($\Gamma = 0$).



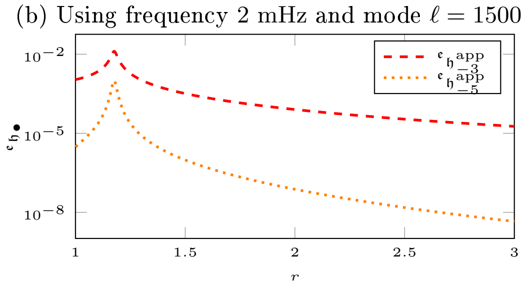
(a) Using frequency 2 mHz and mode $\ell = 10$.



(b) Using frequency 2 mHz and mode $\ell = 1500$.



(c) Evolution of error at 2 mHz, for mode $\ell = 10$.

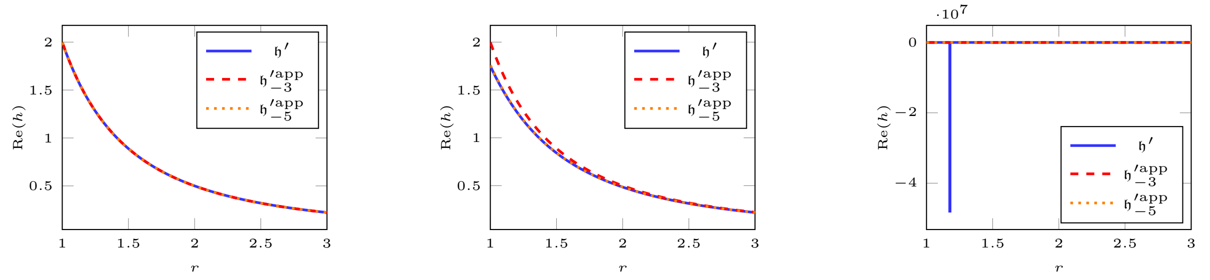


(d) Evolution of error at 2 mHz, for mode $\ell = 1500$.

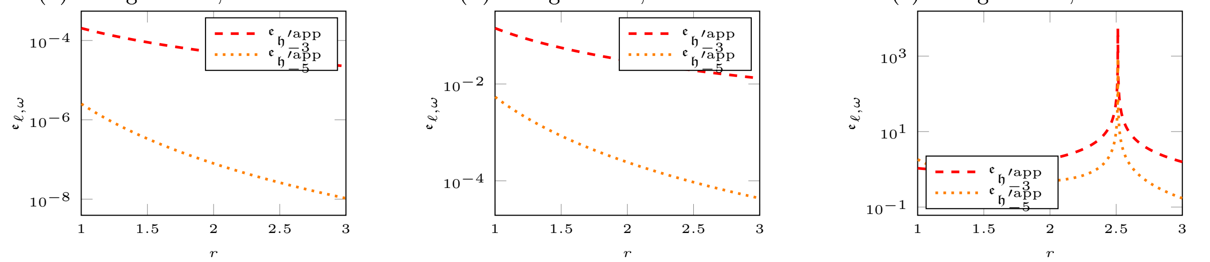
Figure 12: Evaluation of the asymptotic of $\mathfrak{h}$ at order 3 and 5, given by (8.46a) and (8.56) in the case with attenuation $\Gamma/(2\pi) = 20$ µHz.

10^{-12} for $\mathfrak{h}$. At mode $\ell = 250$, the relative error is still acceptable (10^{-4} with order 5 approximation), but the error is particularly high for high modes, as illustrated with mode $\ell = 2500$ in Figures 13 and 14, with, namely, 100% relative error.

Therefore, the asymptotic expansion of $\mathfrak{h}'$ is less accurate than that for $\mathfrak{h}$. The order 3 approximation seems effective only for low modes, while the order 5 appears effective for modes up to a few hundreds. Nonetheless, for very high modes, such that we have illustrated with $\ell = 1500$, one needs an expansion of higher order.

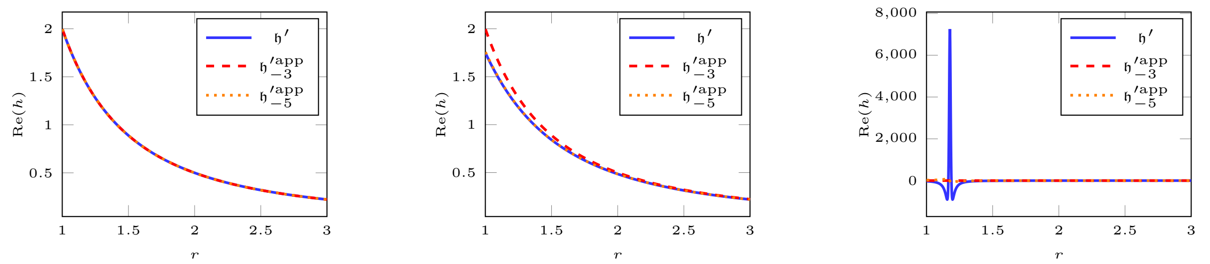


(a) Using 2 mHz, $\ell = 10$. (b) Using 2 mHz, $\ell = 250$. (c) Using 2 mHz, $\ell = 1500$.

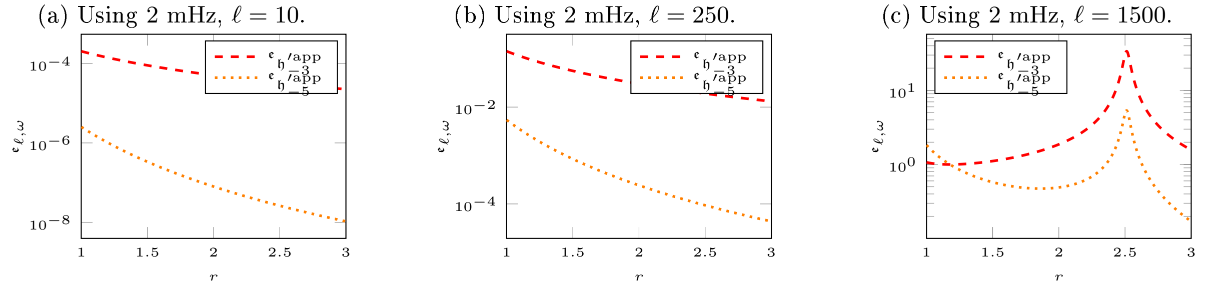


(d) Using 2 mHz, $\ell = 10$. (e) Using 2 mHz, $\ell = 250$. (f) Using 2 mHz, $\ell = 1500$.

Figure 13: Evaluation of the asymptotic of $\mathfrak{h}'$ at order 3 and 5, given by (8.46b) and (8.59) in the case without attenuation ($\Gamma = 0$).



(a) Using 2 mHz, $\ell = 10$. (b) Using 2 mHz, $\ell = 250$. (c) Using 2 mHz, $\ell = 1500$.



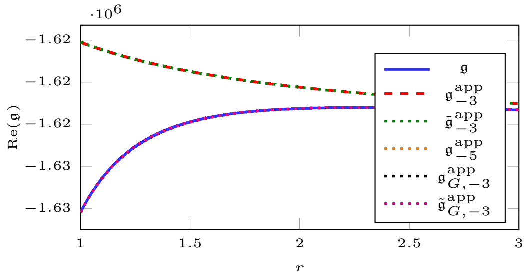
(d) Using 2 mHz, $\ell = 10$. (e) Using 2 mHz, $\ell = 250$. (f) Using 2 mHz, $\ell = 1500$.

Figure 14: Evaluation of the asymptotic of $\mathfrak{h}'$ at order 3 and 5, given by (8.46b) and (8.59) in the case with attenuation $\Gamma/(2\pi) = 20$ µHz.

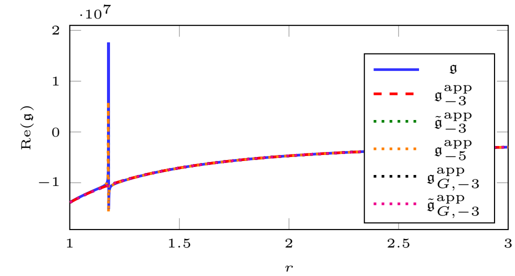
8.3.3 Numerical illustration for the asymptotic of $\mathfrak{g}$

For the asymptotic of $\mathfrak{g}$, we have provided the expansion at order 3 and 5, (8.47a) and (8.60), as well as a variation for the order 3, $\tilde{\mathfrak{g}}^{\text{app}}_{-3}$, given in (8.47b). In addition, we have provided the locally enriched version with the gravity term for the order 3, $\mathfrak{g}^{\text{app}}_{G,-3}$ and $\tilde{\mathfrak{g}}^{\text{app}}_{G,-3}$ given in (8.65) and (8.66). We picture the performance of these approximations in Figures 15 and 16, respectively in the case without and with attenuation.

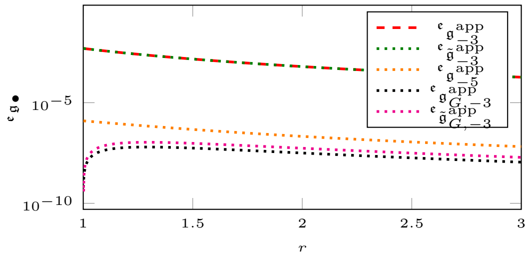
As in the previous comparisons, we do not observe any difference between the case without and with attenuation in terms of accuracy. Comparing the choice of asymptotic expansions, we have the following



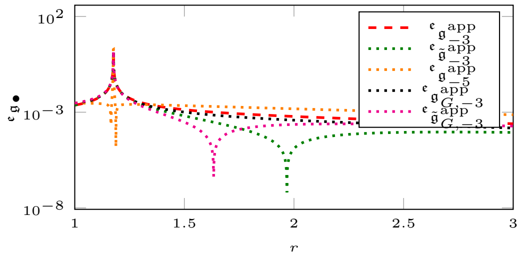
(a) Using frequency 2 mHz and mode $\ell = 10$.



(b) Using frequency 2 mHz and mode $\ell = 1500$.

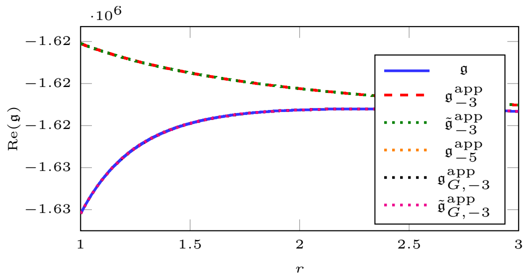


(c) Using frequency 2 mHz and mode $\ell = 10$.

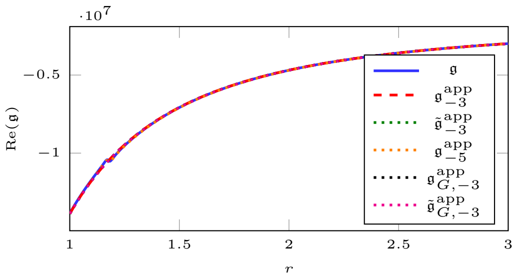


(d) Using frequency 2 mHz and mode $\ell = 1500$.

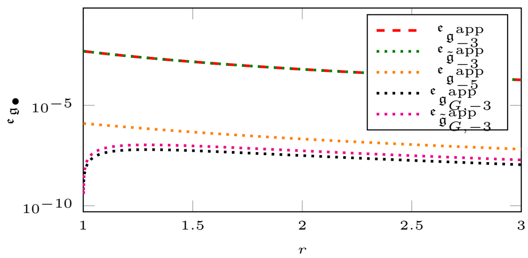
Figure 15: Evaluation of the asymptotic of $\mathfrak{g}$, in the case without attenuation ($\Gamma = 0$).



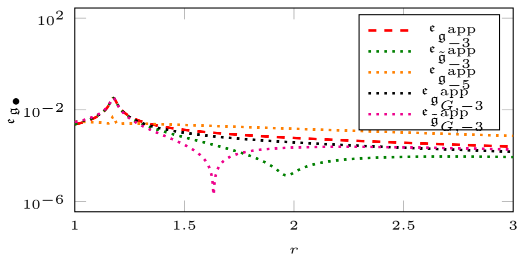
(a) Using frequency 2 mHz and mode $\ell = 10$.



(b) Using frequency 2 mHz and mode $\ell = 1500$.



(c) Using frequency 2 mHz and mode $\ell = 10$.



(d) Using frequency 2 mHz and mode $\ell = 1500$.

Figure 16: Evaluation of the asymptotic of $\mathfrak{g}$, in the case with attenuation $\Gamma/(2\pi) = 20$ µHz.

comments.

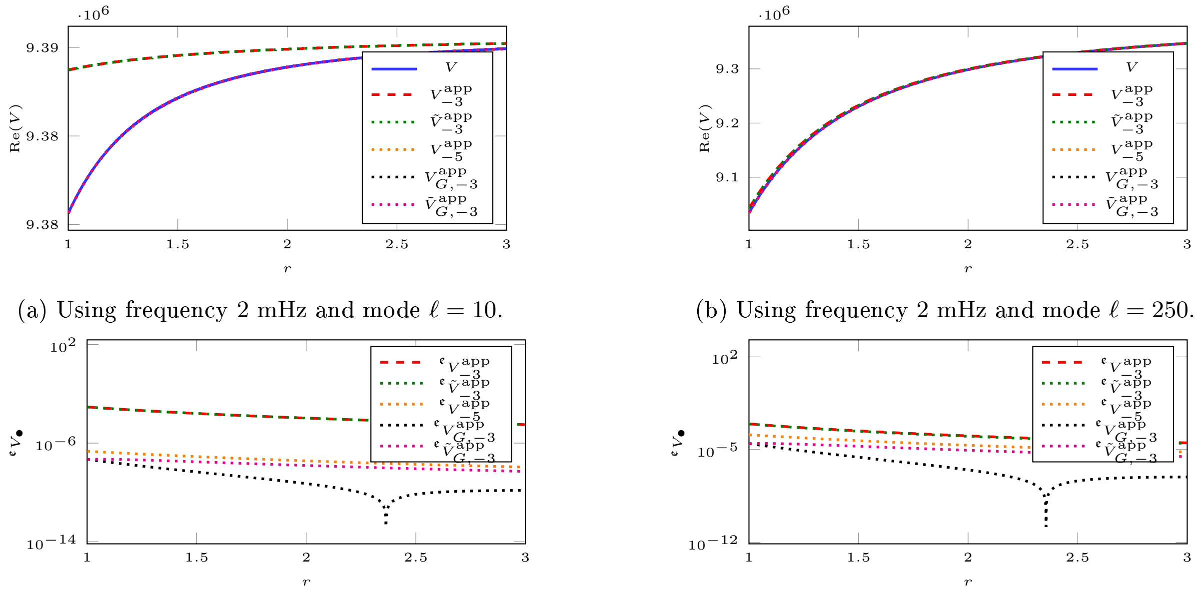
- The order 3 approximation $\mathfrak{g}^{\text{app}}_{-3}$ and $\tilde{\mathfrak{g}}^{\text{app}}_{-3}$ gives the same results at low mode, where we cannot distinguish between the two. On the other hand, at higher modes, $\tilde{\mathfrak{g}}^{\text{app}}_{-3}$ gives better results than $\mathfrak{g}^{\text{app}}_{-3}$.
- Similarly, the enriched versions of the order 3 approximations, $\mathfrak{g}^{\text{app}}_{G,-3}$ and $\tilde{\mathfrak{g}}^{\text{app}}_{G,-3}$ are relatively similar at low mode, but $\tilde{\mathfrak{g}}^{\text{app}}_{G,-3}$ behaves better at high modes.
- We clearly observe the improvement obtained from enriched versions with gravity, $\mathfrak{g}^{\text{app}}_{G,-3}$ and $\tilde{\mathfrak{g}}^{\text{app}}_{G,-3}$, compared to the original version of $\mathfrak{g}^{\text{app}}_{-3}$ and $\tilde{\mathfrak{g}}^{\text{app}}_{-3}$. In particular, the enriched version as as accurate

as the order 5 approximation, $\mathfrak{g}^{\text{app}}_{-5}$.

Overall, we see that the ‘tilda’ versions of the expansion is more efficient, and that the enrichment of the order 3 with the gravity is as (or even more) accurate than the order 5.

8.3.4 Numerical illustration for the asymptotic of V_ℓ

We can finally plot the asymptotics of the potential V, which definitions coincide with the choice of expansion for $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{h}'$. Therefore, we have the order 3 and 5 (V^{app}_{-3} and V^{app}_{-5}) and, for the former, the variation $\tilde{V}^{\text{app}}_{-3}$ and the possibility to enrich with the gravity term: $V^{\text{app}}_{G,-3}$ and $\tilde{V}^{\text{app}}_{G,-3}$. We compare in Figures 17 and 18 in the absence or presence of attenuation respectively.



(a) Using frequency 2 mHz and mode $\ell = 10$.

(b) Using frequency 2 mHz and mode $\ell = 250$.

(c) Using frequency 2 mHz and mode $\ell = 10$.

(d) Using frequency 2 mHz and mode $\ell = 250$.

Figure 17: Evaluation of the asymptotic of V, in the case without attenuation ($\Gamma = 0$).

The results for the potential coincide with the observations given above for the different functions. In particular, the accuracy of the approximation using order 5 can be attained by using the order 3 with an enriched gravity term. While we show the results using a maximal mode of $\ell = 250$, it is clear that using high modes (such as $\ell = 2500$) results in a drastic increase of error, as illustrated in the approximation of $\mathfrak{h}'$.

9 Existence of solutions

With the background quantities given by the `AtmoCAI` model for $r \geq \mathsf{r}_a$, we consider

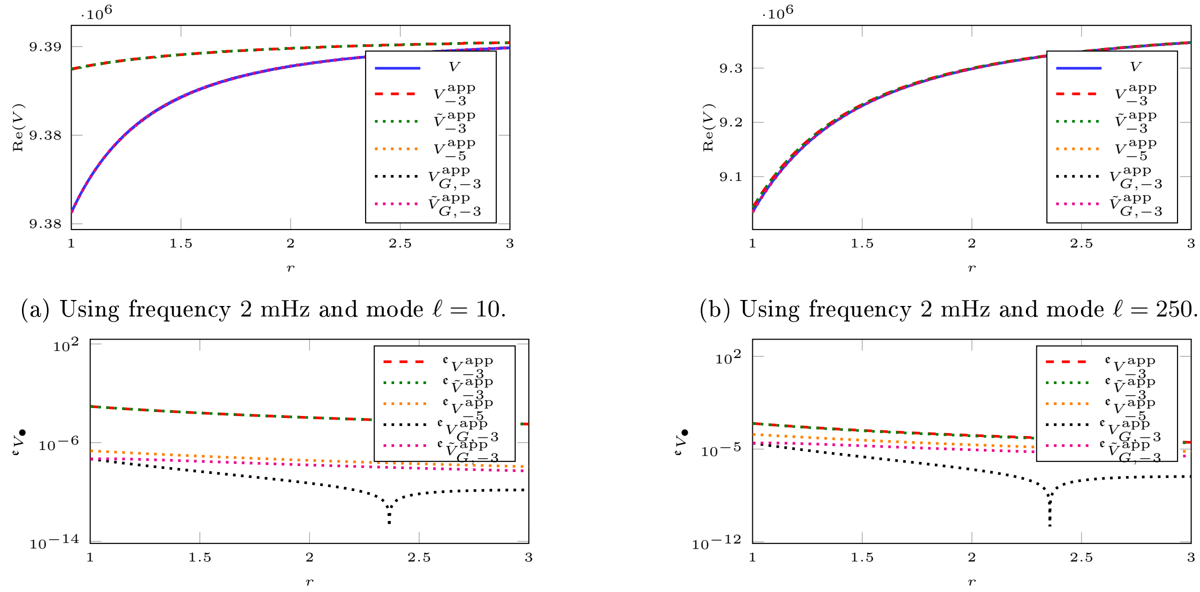
$$-\rho_0 \left(\omega^2 + 2\mathrm{i}\,\omega\,\Gamma\right) \boldsymbol{\xi} + \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) + \rho_0(\boldsymbol{\xi}\cdot\nabla)\nabla\Phi_0 = \mathbf{f} \quad \text{in } \mathbb{R}^3 . \tag{9.1}$$

In this section, we construct a Green's operator, denoted by $\boldsymbol{\mathcal{G}}$ whose Schwartz kernel is given by Green's tensor $\mathbb{G}$ written in a basis made up of vectorial harmonic spherical $\mathbf{P}^m_\ell$, $\mathbf{B}^m_\ell$ and $\mathbf{C}^m_\ell$, so that for a compactly supported smooth vector-valued function $\mathbf{f}$,

$$\boldsymbol{\mathcal{L}}\,\boldsymbol{\mathcal{G}}\,\mathbf{f} = \mathbf{f} , \qquad \mathbf{f} \in \mathcal{D}(\mathbb{R}^3)^3 . \tag{9.2}$$

The component of the 3D Green's kernel $\mathbb{G}$ depends on the radial modal Green's kernel of the modal radial ODE,

$$\big(\hat{q}(r)\,\partial_r^2 + q(r)\,\partial_r + \tilde{q}(r)\big)\, G_\ell(r,s) = \delta(r-s) . \tag{9.3}$$



(a) Using frequency 2 mHz and mode $\ell = 10$.

(b) Using frequency 2 mHz and mode $\ell = 250$.

(c) Using frequency 2 mHz and mode $\ell = 10$.

(d) Using frequency 2 mHz and mode $\ell = 250$.

Figure 18: Evaluation of the asymptotic of V, in the case with attenuation $\Gamma/(2\pi) = 20$ µHz.

This is equivalent to constructing a conjugate modal Green's kernel $\tilde{G}_\ell$,

$$\big(\partial_r^2 \,-\, V_\ell\big)\tilde{G}_\ell \,=\, \delta(r-s)\,. \tag{9.4}$$

The second serves in particular for choosing an appropriate outgoing condition at infinity. Since Green's kernels of ODE are obtained from two homogeneous solutions, we first construct 'regular' homogeneous solutions in Subsection 9.1.1, and then outgoing homogeneous solutions in Subsection 9.1.2. With the construction G_ℓ, the tensor $\mathbb{G}$ is determined *uniquely* $\mathbb{G}$ in Subsection 9.2.

9.1 Existence results for modal radial ODEs

9.1.1 Existence of a regular solution ψ

We have identified the singular points of the coefficients of ODE (7.1) in Table 1. Apart these points, the coefficients of the two ODEs are continuous. To facilitate the construction of solution in the neighborhood of singular points, in particular to apply ODE theory for regular singular points, we assume the following.

Assumption 7 (Analytic background assumption)**.** *We assume that* c_0*,* ρ_0 *and* γ *are analytic in a small neigbhorhood of each point in the singular set* S*, and outside of which they are continuous, with*

$$\Sigma_{reg\ sing} \,=\, \{0\} \qquad\qquad \text{for } \Gamma > 0\,, \tag{9.5a}$$

$$\Sigma_{reg\ sing} \,=\, \{0\,,\, r^\star_{\mathrm{i},\omega,\ell}\,,\, r^\star_{a1,\omega}\,,\, r^\star_{a2,\omega,\ell}\} \qquad \text{for } \Gamma = 0\,. \tag{9.5b}$$

Under Assumption 7, we can construct a regular solution $\phi = \phi_\ell$ of the original radial modal ODE (7.1),

$$\Big(\hat{q}(r)\,\partial_r^2 \,+\, q(r)\,\partial_r \,+\, \tilde{q}(r)\Big)\psi \,=\, 0\,. \tag{9.6}$$

This will give a corresponding 'more regular'[4] solution $\psi = \psi_\ell$ for the conjugate radial modal ODE (7.2),

$$\big(-\partial_r^2 \,+\, V(r)\big)\psi \,=\, 0\,, \tag{9.7}$$

via the relation (4.66),

$$\psi_\ell \,=\, e^{-\frac{1}{2}\int^r \mathfrak{h}_\ell}\,\phi_\ell\,. \tag{9.8}$$

[4] We recall that $\tilde{\lambda}^+ = \lambda^+ - 2$ for $\ell > 0$ and $\tilde{\lambda}^+ = \lambda^+ - 1 = 0$ for $\ell = 0$.

Regular solution for the original radial ODE For $\Gamma > 0$, we recall that the only singular point is at $r = 0$ with the indicial roots given by

$$\lambda_0^+ \,=\, \begin{cases} \ell - 1 & ,\, \ell > 0 \\ 1 & ,\, \ell = 0 \end{cases}, \qquad \lambda_0^- \,=\, \begin{cases} -\ell - 2 & ,\, \ell > 0 \\ -2 & ,\, \ell = 0 \end{cases}. \tag{9.9}$$

Assume also the analyticity neighborhood of the background at the point is $(-\delta, \delta)$, with $\delta > 0$. We apply Theorem 3 for the regular singular ODE in order to construct a regular solution ψ_ℓ on $[0, \delta)$ satisfying,

$$\lim_{r\to 0} r^{-\lambda_0^+}\, \psi_\ell \,=\, 1\,. \tag{9.10}$$

This solution $\psi_\ell|_{[0,\delta)}$ is then extended to $[0, \infty)$ by using Theorem 2 applied to the interval $[-\delta/2, \infty)$.

For $\Gamma = 0$, we have shown that the set of regular singular points contains more than just $r = 0$, cf. (9.5). Once we make a choice of a solution at $r = 0$ and we can only extend this solution without making further choice of indicial exponents at the remaining singular points ($r^\star_{\mathrm{i},\omega,\ell}$, $r^\star_{\mathrm{a1},\omega}$, and $r^\star_{\mathrm{a2},\omega,\ell}$). The extension is obtained by using Theorem 2 on the interval where the coefficients of (7.1) are continuous, and using Theorem 3 to extend after a regular singular point.

Remark 17. *When we extend past a regular singular point, what we simply do is to determine the highest order term c_0 and $\tilde{c}_0$ in Theorem 3, and the extended solution is a linear combination of the u_1 and u_2. Luckily, the indicial exponents at the nonzero singular points for the ODE (7.1) are all non-negative, cf. Table 1.* △

9.1.2 Existence of outgoing homogeneous solution

From the results of Section 7 and Subsection 9.1.1, there exists $r_{\mathrm{reg}} > 0$ such that V_ℓ is bounded on $[r_{\mathrm{reg}}, \infty)$, cf. Proposition 19. V_ℓ has the asymptotic expansion at infinity,

$$V_\ell(r) \,=\, -\mathsf{k}^2 - \frac{\alpha_{\mathrm{ad}}}{r} + \frac{\mu_\ell^2 - \frac{1}{4}}{r^2} + \mathcal{O}(r^{-3})\,. \tag{9.11}$$

Here, μ_ℓ depends on v_{-2} defined in Proposition 19 and is explicitely given in (9.40), and we use the notations

$$\mathsf{k}^2 \,:=\, \mathsf{v}_0 \,=\, \frac{\sigma^2}{\mathsf{c}_0^2} - \frac{\alpha^2}{4}\,, \qquad \mathsf{k} \,:=\, \sqrt{\mathsf{k}^2}\,; \tag{9.12a}$$

$$\alpha_{\mathrm{ad}} \,=\, -\mathsf{v}_{-1} \,=\, \frac{\alpha}{\gamma}(2-\gamma)\,, \tag{9.12b}$$

where $\sqrt{\ }$ uses the Argument branch $[0, 2\pi)$. Under the physical assumption of (2.1), we have

$$\alpha_{\mathrm{ad}} \,>\, 0\,. \tag{9.13}$$

In another word, the function

$$\mathbb{R}_+ \ni r \quad\mapsto\quad V_\ell(r + r_{\mathrm{reg}})\,, \tag{9.14}$$

is smooth and inherits the same asymptotic with V_ℓ at infinity. To focus on the behavior of the solution to (7.2),

$$\big(-\partial_r^2 + V_\ell(r)\big)\, \tilde{a}_\ell^m \,=\, 0\,, \tag{9.15}$$

at infinity, we follow [2, Eqn 3.4 - 3.6] to shift the problem to $(r_{\mathrm{reg}}, \infty)$, on which V_ℓ is bounded.

Shifted problem Consider a solution w defined by

$$w(r) \,=\, \tilde{a}(r + r_{\mathrm{reg}})\,, \tag{9.16}$$

then w satisfies

$$-\frac{d^2}{dr^2}\, w + V_{\mathrm{shifted}}\, w \,=\, 0\,, \qquad V_{\mathrm{shifted}}(r) \,=\, V(r + r_{\mathrm{reg}})\,, \qquad r > 0\,. \tag{9.17}$$

V_{shifted} is a complex-valued function which is bounded, smooth and has asymptotic,

$$V_{\text{shifted}}(r) \,=\, -\mathsf{k}^2 \,-\, \frac{\alpha_{\text{ad}}}{r+r_{\text{reg}}} \,+\, \frac{\mu_\ell^2-\frac{1}{4}}{(r+r_{\text{reg}})^2} \,+\, \mathsf{O}((r+r_{\text{reg}})^{-3})\,. \tag{9.18}$$

We define

$$q_{\text{L}}(r) \,:=\, -\frac{\alpha_{\text{ad}}}{r+r_{\text{reg}}}\,; \tag{9.19a}$$

$$q_{\text{S}}(r) \,:=\, V_{\text{shifted}}(r) \,-\, q_{\text{L}}(r) \,+\, \mathsf{k}^2 \,=\, \frac{\mu_\ell^2-\frac{1}{4}}{(r+r_{\text{reg}})^2} \,+\, \mathsf{O}(r^{-3})\,. \tag{9.19b}$$

Then ODE (9.17) on $r > r_{\text{reg}}$ is written as

$$\boxed{\left(-\frac{d^2}{dr^2} \,-\, \lambda^2 \,+\, q_{\text{S}}(r) \,+\, q_{\text{L}}(r)\right) w \,=\, 0\,, \qquad r>0\,.} \tag{9.20}$$

In our case, $\lambda^2 = \mathsf{k}^2$ and q also depends on k.

Outgoing solution on $[r_{\mathbf{reg}},\infty)$ The ODE (9.20) is in the form of Equation (2.1) of [2], and the potential q satisfies hypothesis (H3) of Proposition 2.1 and Theorem 2.2 in [2]. We first list this hypothesis,

$$q(r) \,=\, q_{\text{S}} \,+\, q_{\text{L}} \quad \text{is a complex-valued function}\,; \tag{9.21a}$$

$$q_{\text{S}} \,\in\, L^1(\mathbb{R}_+)\,; \tag{9.21b}$$

$$q_{\text{L}} \,\in\, \mathcal{C}^2(\mathbb{R}_+)\,; \tag{9.21c}$$

$$\lim_{r\to\infty} q_{\text{L}} \,=\, 0 \qquad , \qquad \partial_r^j q \,=\, \mathsf{O}(r^{-j/2-\epsilon})\,,\, r\to\infty\,,\, j=1,2 \,\text{and}\, \epsilon>0\,. \tag{9.21d}$$

Proposition 2.1 of [2] allows us to construct a global phase φ having the property: for all $j \in \mathbb{N}$, there exists an analytic function $g_j(\lambda)$ on $\{\lambda : \lambda^2 > 1/j\}$,

$$\varphi(r) = \int_0^r \sqrt{\lambda \,-\, \left(1 \,-\, \chi(\tfrac{r}{j})\right)\, q_{\text{L}}(s)}\, ds \,+\, g_j(\lambda)\,, \tag{9.22}$$

such that

$$\lambda \mapsto \varphi(r,\lambda) \quad \text{is analytic in } \lambda \in \mathbb{C}\setminus\{0\}\,, \tag{9.23}$$

to define incoming/outgoing solution. Here, the cut-off function χ and the sequence χ_j are defined as

$$\chi \in \mathcal{C}^\infty(\mathbb{R})\,, \qquad \chi(r) = \begin{cases} 1 & |r| \leq 0.5 \\ 0 & |r| \geq 1 \end{cases}\,, \qquad \chi_j(r) \,:=\, \chi\left(\tfrac{r}{j}\right)\,. \tag{9.24}$$

Note that the analytic function $g_j(\lambda)$ only depends on λ.

Remark 18. *The phase function $\varphi = \varphi(r,\mathsf{k})$ can be chosen to be an exact or approximate solution to the eikonal equation, cf. [31, Eqn. 0.15],*

$$|\varphi'(r,\mathsf{k})|^2 \,+\, \mathsf{q}_{\text{L}} \,=\, \mathsf{k}^2\,, \tag{9.25}$$

hence,

$$|\varphi'(r,\mathsf{k})|^2 \,+\, \frac{-\alpha_{\text{ad}}}{r} \,=\, \mathsf{k}^2 \quad \Rightarrow \quad \varphi(r,\mathsf{k}) \,=\, \int_0^r \sqrt{\mathsf{k}^2 \,-\, \frac{\alpha_{\text{ad}}}{s}}\, ds \,=\, \mathsf{k}\int_{r_0}^r \sqrt{1 \,+\, \frac{\alpha_{\text{ad}}}{s\,\mathsf{k}^2}}\, ds\,. \tag{9.26}$$

We obtain,

$$\varphi(r,\mathsf{k}) \,\sim\, \mathsf{k}\int_{r_0}^r \left(1 \,-\, \frac{\alpha_{\text{ad}}}{2\,s\,\mathsf{k}^2}\right) ds \,\sim\, \mathsf{k}\,r \,+\, \frac{\alpha_{\text{ad}}}{2\,\mathsf{k}}\,\log r\,. \tag{9.27}$$

△

We next apply Theorem 2.2 of [2], which gives the unique existence of the outgoing solution to (9.17), denoted by w_+. We list the stronger results for real q_{L}. We define,

$$\mathbb{C}_\pm \;:=\; \{z \in \mathbb{C} \,\big|\, \pm \operatorname{Im} z \,>\, 0\}\,. \tag{9.28}$$

Theorem 1 (Theorem 2.2 of [2])**.** *Suppose that the potential q satisfies the hypothesis (9.21).*

1. *For each $\sqrt{\lambda} \in \mathbb{C}_\pm \setminus \{0\}$, the equation*

$$\big(-\partial_r^2 \,-\, \lambda^2 \,+\, \mathsf{q}\big) w \;=\; 0\,, \qquad r > 0\,, \tag{9.29}$$

has a unique solution $w_\pm(r)$ satisfying the asymptotic relation

$$w_\pm(r,\lambda) \;=\; e^{\pm \mathrm{i}\,\varphi(r,\lambda)}\big(1 \,+\, \mathsf{o}(1)\big)\,, \qquad \textit{as } r \to \infty\,, \tag{9.30}$$

and the mapping

$$\mathbb{C}_\pm \setminus \{0\} \quad \ni \quad \lambda \quad \mapsto \quad \underbrace{w_+(r,\lambda)}_{\textit{defined in } (9.29)} \qquad \textit{is analytic}\,. \tag{9.31}$$

2. *This family extends continuously to*

$$\lambda \,\in\, \overline{\mathbb{C}}_\pm \,\setminus\, \{0\}\,, \tag{9.32}$$

and the asymptotic relation (9.29) holds uniformly in

$$0 \,\le\, \operatorname{Arg}(\pm\lambda) \,\le\, \pi\,, \qquad |\lambda| \,\ge\, \delta \,>\, 0\,. \tag{9.33}$$

As a result of the above theorem, we obtain the solution $w_+(r,\lambda)|_{(0,\infty)}$ to (9.29) which provides the unique solution to (9.15) on $(r_{\mathrm{reg}},\infty)$, which satisfies the asymptotic relation

$$\tilde{\phi}(r,\mathsf{k}) \;=\; e^{\varphi_\lambda(r,\mathsf{k})}\Big(1 \,+\, \mathsf{o}(1)\Big)\,, \qquad \text{as } r \to \infty\,. \tag{9.34}$$

We denote this solution by

$$\tilde{\phi}\big|_{(r_{\mathrm{reg}},\infty)}\,. \tag{9.35}$$

Extension of outgoing solution on $(0,\infty)$ We next extend the solution constructed in (9.35) to a solution to (9.15) on $(0,\infty)$. This result is instant for the case where $\Gamma > 0$, since $r_{\mathrm{reg}} > 0$, cf. Table 1. In the case where $\Gamma = 0$, the solution is extended backward up to $r = 0$, using Theorem 2, on the interval where the coefficients of (7.1) are continuous, and using Theorem 3 to extend pass a regular singular point. The solution remains continuous on $(0,\infty)$, due to the fact that the indicial exponents of non-zero regular points are positive, see also Remark 17. In short, we obtain an outgoing solution $\tilde{\phi}_\ell$ to (7.1) on $(0,\infty)$ that is bounded on a compact subset of $(0,\infty)$, and that has

$$\tilde{\phi}(r,\mathsf{k}) \;=\; e^{\varphi_\lambda(r,\mathsf{k})}\Big(1 \,+\, \mathsf{o}(1)\Big)\,, \qquad \text{as } r \to \infty\,. \tag{9.36}$$

9.1.3 Approximate outgoing modal coefficient in the atmosphere via Whittaker equation

We restrict ourselves to the atmosphere, and consider the ODE with the potential that takes into account the first three terms in the asymptotic expansion of V_ℓ at infinity.

We call

$$\boxed{\left(-\partial_r^2 \,-\, \mathsf{k}^2 \,-\, \frac{\alpha_{\mathrm{ad}}}{r} \,+\, \frac{\mathsf{V}_{-2}}{r^2}\right)\hat{a} \;=\; 0\,,} \tag{9.37}$$

the *approximate conjugate radial equation.* We also define

$$Q \;:=\; \mathsf{k}^2 \,+\, \frac{\alpha_{\mathrm{ad}}}{r} \,-\, \frac{V_{-2}}{r^2}\,. \tag{9.38}$$

Normalization of approximate radial equation (9.37) to the Whittaker equation Introduce the change of unknown

$$z := 2\,e^{\mathrm{i}\frac{\pi}{2}}\,\mathsf{k}\,r\,. \tag{9.39}$$

Then for $\tilde{A}(z)$ defined as $\tilde{a}(r) = \tilde{A}(z := 2\mathrm{i}\mathsf{k}r)$, we have

$$\partial_r \hat{a} \,=\, 2\,\mathrm{i}\,\mathsf{k}\,\partial_z \hat{A} \quad \Rightarrow -\partial_r^2 \hat{a} \,=\, 4\,\mathsf{k}^2 \partial_z^2 \hat{A}\,.$$

We divide both sides of (9.37) by $4\mathsf{k}^2$ to get,

$$-\frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{4\mathsf{k}^2} \,=\, -\mathrm{i}\,\frac{\alpha_{\mathrm{ad}}}{2\mathsf{k}}\,\frac{1}{z} \quad , \quad \frac{\mathsf{v}_{-2}}{r^2\,4\mathsf{k}^2} \,=\, -\frac{\mathsf{v}_{-2}}{z^2}\,.$$

Arising from this calculation are μ_ℓ and η defined as

$$\eta_{\mathrm{ad}} \,:=\, \frac{\alpha_{\mathrm{ad}}}{2\,\mathsf{k}}\,; \tag{9.40a}$$

$$\frac{1}{4} \,-\, \mu_\ell^2 \,:=\, -\mathsf{v}_{-2} \,=\, -2 \,-\, \ell(\ell+1)\left(1 \,-\, \frac{\alpha^2}{k_0^2}\frac{\gamma\,-\,1}{\gamma^2}\right)\,. \tag{9.40b}$$

In the new notations, the approximate conjugate ODE (9.37) is written as,

$$\boxed{\left(-\frac{d^2}{dr^2} \,-\, \mathsf{k}^2 \,+\, \frac{(-\alpha_{\mathrm{ad}})}{r} \,+\, \frac{\mu_\ell^2 \,-\, \frac{1}{4}}{r^2}\right)\hat{a} \,=\, 0\,.} \tag{9.41}$$

The truncated potential Q defined in (9.38) is

$$Q \,=\, \mathsf{k}^2 \,+\, \frac{\alpha_{\mathrm{ad}}}{r} \,-\, \frac{\mu_\ell^2 \,-\, \frac{1}{4}}{r^2}\,. \tag{9.42}$$

In particular, the new variable $\hat{A}$ solves the Whittaker equation,

$$\boxed{\partial_z^2 \hat{A} \,+\, \left(-\frac{1}{4} \,+\, \frac{-\mathrm{i}\,\eta_{\mathrm{ad}}}{r} \,+\, \frac{\frac{1}{4} \,-\, \mu_\ell^2}{z^2}\right)\hat{A} \,=\, 0\,.} \tag{9.43}$$

Since a pair of linearly independent solutions to (9.43) in a neighborhood of infinity is given by

$$\underset{\text{incoming}}{\mathrm{W}_{-\mathrm{i}\,\eta_{\mathrm{ad}}\,,\,\mu_\ell}(z)}\,, \qquad \underset{\text{outgoing}}{\mathrm{W}_{\mathrm{i}\,\eta_{\mathrm{ad}}\,,\,\mu_\ell}(e^{-\mathrm{i}\,\pi}\,z)}\,, \tag{9.44}$$

the corresponding pair for (9.41) is

$$\underset{\text{incoming}}{\mathrm{W}_{-\mathrm{i}\,\eta_{\mathrm{ad}}\,,\,\mu_\ell}(2\,\mathrm{i}\,\mathsf{k}\,r)}\,, \qquad \underset{\text{outgoing}}{\mathrm{W}_{\mathrm{i}\,\eta_{\mathrm{ad}}\,,\,\mu_\ell}(e^{-\mathrm{i}\,\pi}\,2\mathrm{i}\,\mathsf{k}\,r)}\,. \tag{9.45}$$

Approximate solution An outgoing solution for $\tilde{a}$ can be approximated by the Whittaker function

$$\tilde{a}_\ell^m \,\sim\, \mathrm{W}_{\mathrm{i}\eta_{\mathrm{ad}}\,,\,\mu}(-2\,\mathrm{i}\,\mathsf{k}r)\,. \tag{9.46}$$

Remark 19. *The key here is that*

$$\boxed{\lim_{r\to\infty} \partial_r\,\tilde{a}(r) \,-\, \mathrm{i}\,\mathsf{k}\,\tilde{a}(r) \,=\, 0\,,} \tag{9.47}$$

such that $a = a_\ell^m$ satisfies to

$$\boxed{\lim_{r\to\infty} \partial_r a(r) \,-\, \frac{\mathfrak{h}(r)}{2}a(r) \,-\, \mathrm{i}\,\mathsf{k}\,a \,=\, 0\,.} \tag{9.48}$$

9.1.4 Modal Green kernel

We work under Assumption 7. We now patch together the regular and outgoing solutions constructed in Subsection 9.1.1 and Subsection 9.1.2 respectively to obtain the radial modal Green's function.We can either build the Green's kernel for the original radial ODE directly or first construct one for the conjugate ODE and then obtain that for the original ODE. We discuss both approaches below.

Approach 1 We construct directly the Green's kernel of

$$\Big(\hat{q}(r)\,\partial_r^2 a \,+\, q(r)\,\partial_r a \,+\, \tilde{q}(r)\Big) G_\ell(r,s) \;=\; \delta(r-s) \tag{9.49}$$

where δ denotes the Dirac function. We have

$$\boxed{G_\ell(r,s) \;:=\; -\frac{\mathrm{H}(s-r)\,\phi(r)\,\tilde{\phi}(s) \,+\, \mathrm{H}(r-s)\,\tilde{\phi}(r)\,\phi(s)}{\mathcal{W}(\phi,\tilde{\phi})(s)}\,,} \tag{9.50}$$

where H is the Heavyside function and using the two homogeneous solutions ϕ, $\tilde{\phi}$,

$$\begin{aligned} \Big(\hat{q}(r)\,\partial_r^2 \,+\, q(r)\,\partial_r \,+\, \tilde{q}(r)\Big)\phi \;&=\; 0 \qquad \text{on } (0,\infty)\,; \\ \Big(\hat{q}(r)\,\partial_r^2 \,+\, q(r)\,\partial_r \,+\, \tilde{q}(r)\Big)\tilde{\phi} \;&=\; 0 \qquad \text{on } (0,\infty)\,. \end{aligned} \tag{9.51}$$

In the above expression, $s \mapsto \mathcal{W}(\phi,\tilde{\phi})(s)$ is the Wronskian of $\phi(s)$ and $\tilde{\phi}(s)$. The solution ϕ is the unique regular one on $(0,\infty)$ to (9.51), satisfying

$$\lim_{r\to\lambda_+} r^{-\lambda_0^+}\,\phi_\ell(r) \;=\; 1\,. \tag{9.52}$$

The indicial exponent λ_0^+ is given in (7.30) and (7.31). On the other hand, $\tilde{\phi}$ is obtain from the unique outgoing homogeneous solution $\tilde{\psi}$ to the conjugate ODE defined in (9.57),

$$\tilde{\phi}(r) \;:=\; e^{\frac{1}{2}\int^r \mathfrak{h}}\,\tilde{\psi}(r)\,. \tag{9.53}$$

Thus, $\tilde{\phi}$ satisfies the asymptotic relation

$$\tilde{\phi}(r) \;=\; e^{\frac{1}{2}\int^r \mathfrak{h}}\,e^{\mathrm{i}\,\varphi(r,\mathsf{k})}\,(1 \,+\, \mathsf{o}(1))\,, \qquad r\to\infty\,. \tag{9.54}$$

Approach 2 : In this approach, we first construct a Green's kernel for the conjugate equation, and from there we construct a Green's kernel for the original problem. For homogeneous conjugate problem,

$$\big(-\partial_r^2 \,+\, V(r)\big)\psi \;=\; 0 \tag{9.55}$$

the results of previous sections have shown that, with or without attenuation,

1. There exists a unique regular solution ψ_ℓ on $(0,\infty)$ that satisfies

$$\lim_{r\to\lambda_+} r^{-\tilde{\lambda}_0^+}\,\psi_\ell \;=\; 1\,, \tag{9.56}$$

   with $\tilde{\lambda}_0^+$ given in (7.125),

2. There exists a unique outgoing solution ψ_ℓ on $(0,\infty)$ satisfying

$$\tilde{\psi}_\ell \;=\; e^{\mathrm{i}\,\varphi(r,\mathsf{k})}(1 \,+\, \mathsf{o}(1))\,, \qquad \text{as}\, r\to\infty\,, \tag{9.57}$$

   where the phase function is independent of ℓ, and satisfies asymptotic relation,

$$\varphi(r,\mathsf{k}) \;=\; \mathsf{k}\,r \,+\, -\frac{\alpha_{\mathrm{ad}}}{2\mathsf{k}}\log r \,+\, \mathsf{k}^{-2}\mathsf{o}(1)\,. \tag{9.58}$$

From this results, the outgoing Green's function $\tilde{G}_\ell$ that solves

$$\big(-\partial_r^2 + V_\ell\big)\,\tilde{G}_\ell \,=\, \delta(r-s)\,, \tag{9.59}$$

is given by

$$\boxed{\tilde{G}_\ell(r,s) \,:=\, -\frac{\mathrm{H}(s-r)\psi(r)\,\tilde{\psi}(s) \,+\, \mathrm{H}(r-s)\tilde{\psi}(r)\,\psi(s)}{\mathcal{W}(\psi,\tilde{\psi})(s)}\,.} \tag{9.60}$$

From the transformation of the right-hand side between the original ODE and the conjugate ODE, cf., (4.64) and (4.65), the outgoing modal Green's function G_ℓ (9.49) is obtained by,

$$\boxed{G_\ell(r,s) \,=\, -e^{\frac{1}{2}\int^r \mathfrak{h}}\, e^{-\frac{1}{2}\int^s \mathfrak{h}}\, \frac{\tilde{G}_\ell(r,s)}{\hat{q}(s)}\,.} \tag{9.61}$$

9.2 Outgoing solution for 3D Green's kernel

Characterization of outgoing solutions As before, we work under Assumption 7. We now return to the 3D equation (9.1). From the results of Proposition 3 and (4.26), we see that the solution of the vectorial equation (9.1) is uniquely determined by its radial component. In particular, using convention in Remark 7, the solution $\boldsymbol{\xi}$ to (9.1),

$$-\rho_0\left(\omega^2 \,+\, 2\mathrm{i}\,\omega\,\Gamma\right)\boldsymbol{\xi} \,+\, \boldsymbol{\mathcal{P}}(\boldsymbol{\xi}) \,+\, \rho_0(\boldsymbol{\xi}\cdot\nabla)\nabla\Phi_0 \,=\, \mathbf{f} \quad \text{in } \mathbb{R}^3\,, \tag{9.62}$$

with a compactly supported right-hand side

$$\mathbf{f} \,=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} f_\ell^m(r)\,\mathbf{P}_\ell^m(\widehat{\mathbf{x}}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} g_\ell^m(r)\,\mathbf{B}_\ell^m(\widehat{\mathbf{x}}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} h_\ell^m(r)\,\mathbf{C}_\ell^m(\widehat{\mathbf{x}})\,, \tag{9.63}$$

is given by

$$\boldsymbol{\xi} \,=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(r)\,\mathbf{P}_\ell^m(\widehat{\mathbf{x}}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} b_\ell^m(r)\,\mathbf{B}_\ell^m(\widehat{\mathbf{x}}) \,+\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} c_\ell^m(r)\,\mathbf{C}_\ell^m(\widehat{\mathbf{x}})\,. \tag{9.64}$$

The radial coefficients are given by, cf. (4.66),

$$a_\ell^m \,=\, \int_0^\infty G_\ell(r,s)\,\mathfrak{f}_\ell^m(s)\,ds\,; \tag{9.65a}$$

$$\text{where} \quad \mathfrak{f}_\ell^m \,=\, -\frac{C_{12}}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}} \,-\, \frac{\ell(\ell+1)}{r}\partial_r\left(\frac{1}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}}\right) \,+\, \frac{f_\ell^m}{\gamma\,\mathrm{p}_0}\,. \tag{9.65b}$$

with G_ℓ the physical kernel constructed in (9.50) (or equivalently (9.61)), and the horizontal ones by, cf. (4.45),

$$\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \,=\, \frac{1}{r}\frac{1}{C_{22}}\partial_r a_\ell^m \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)}\right)\frac{1}{rC_{22}}a_\ell^m \,+\, \frac{1}{C_{22}}\,\frac{g_\ell^m}{\gamma\,\mathrm{p}_0\,\sqrt{\ell(\ell+1)}}\,, \tag{9.66}$$

and

$$c_\ell^m \,=\, \frac{h_\ell^m(r)}{-\sigma^2\rho_0 \,+\, \frac{\mathrm{p}_0' + \rho_0\Phi_0'}{r}}\,. \tag{9.67}$$

Note that the expression for c_ℓ^m is simplified to $c_\ell^m \,=\, \frac{h_\ell^m(r)}{-\sigma^2\rho_0}$ for the interior and

$$\frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)} \,=\, \begin{cases} \frac{\alpha}{\gamma} & r \geq r_a \\ -\frac{\Phi_0'(r)}{\mathrm{c}_0^2(r)} & r \leq r_a\,, \qquad \text{cf. (6.28a)} \end{cases}\,. \tag{9.68}$$

Since the phase function φ in (9.58) is constructed to be independent of ℓ, the characterization (9.57) remains the same for all level of (m,ℓ) and the above solution also satisfies a similar asymptotic relation, which characterizes it as an outgoing solution. In particular, we recall from (9.27) and (9.54) and Proposition 18 that

$$\tilde{\phi}(r) \,=\, e^{\frac{1}{2}\int^r \mathfrak{h}}\, e^{\mathrm{i}\,\varphi(r,\mathsf{k})}\,(1\,+\,\mathsf{o}(1))\,, \qquad r\to\infty\,; \tag{9.69a}$$

$$\varphi(r,\mathsf{k}) \,\sim\, \mathsf{k}\,r \,+\, \frac{\alpha_{\mathrm{ad}}}{2\,\mathsf{k}}\,\log r\,; \tag{9.69b}$$

$$\mathfrak{h} \,=\, \alpha \,-\, \frac{2}{r} \,+\, \mathsf{O}(r^{-3})\,. \tag{9.69c}$$

Thus, we have

$$\tilde{\phi} \,=\, \frac{e^{\frac{1}{2}\alpha r}}{r} e^{\mathrm{i}\,\varphi(r,\mathsf{k})}\,(1\,+\,\mathsf{o}(1))\,, \qquad r\to\infty\,. \tag{9.70}$$

Definition 3. *A solution* $\boldsymbol{\xi}\,\in\,H^2_{loc}(\mathbb{R}^3)$ *to* (9.1) *is called outgoing or physical if its radial part satisfies one of the following asymptotic relation for some continuous function* a*,*

$$\boldsymbol{\xi}\cdot\mathbf{e}_r \,=\, \frac{1}{|\mathbf{x}|}\, e^{\frac{1}{2}\alpha\,|\mathbf{x}|}\, e^{\mathrm{i}\,\varphi(|\mathbf{x}|,\mathsf{k})} \left(a\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) \,+\, \mathsf{o}(1)\right)\,, \qquad \text{as} \quad |\mathbf{x}|\to\infty\,. \tag{9.71}$$

Structure of the 3D outgoing Green's kernel For $\mathbf{x}$ and $\mathbf{s}\in\mathbb{R}^3$, we have written

$$r \,=\, |\mathbf{x}|\,, \qquad s = |\mathbf{s}|\,, \qquad \widehat{\mathbf{x}} \,=\, \tfrac{\mathbf{x}}{|\mathbf{x}|}\,. \tag{9.72}$$

Using the orthogonality of vector spherical harmonics,

$$\int_{\mathbb{S}^2} \mathbf{P}_\ell^m(\widehat{\mathbf{x}})\cdot\overline{\mathbf{P}_{\tilde{\ell}}^{\tilde{m}}(\widehat{\mathbf{x}})}\,d\sigma \,=\, \int_{\mathbb{S}^2} \mathbf{B}_\ell^m(\widehat{\mathbf{x}})\cdot\overline{\mathbf{B}_{\tilde{\ell}}^{\tilde{m}}(\widehat{\mathbf{x}})}\,d\sigma \,=\, \int_{\mathbb{S}^2} \mathbf{C}_\ell^m(\widehat{\mathbf{x}})\cdot\overline{\mathbf{C}_{\tilde{\ell}}^{\tilde{m}}(\widehat{\mathbf{x}})}\,d\sigma \,=\, \delta_{m\tilde{m}}\,\delta_{\ell\tilde{\ell}}\,. \tag{9.73}$$

we can decompose the 3D Green's kernel of $\mathbb{G}$ in the basis of second-order tensors made up from these vector harmonic basis. In particular, we find the scalar distributions acting on the radial direction,

$$A_\ell^m(r,s)\,,\; B_\ell^m(r,s)\,,\; C_\ell^m(r,s)\,,\; D_\ell^m(r,s)\,,\; E_\ell^m(r,s)\,, \tag{9.74}$$

so that

$$\begin{aligned}\mathbb{G}(\mathbf{x},\mathbf{s}) \,=&\; A_0^0(r,s)\,\mathbf{P}_0^0(\widehat{\mathbf{x}})\otimes\mathbf{P}_0^0(\widehat{\mathbf{s}})\\ &+\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell}\Big(A_\ell^m(r,s)\,\mathbf{P}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{P}_\ell^m(\widehat{\mathbf{s}}) \,+\, B_\ell^m(r,s)\,\mathbf{P}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{B}_\ell^m(\widehat{\mathbf{s}})\\ &+\, C_\ell^m(r,s)\mathbf{B}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{P}_\ell^m(\widehat{\mathbf{s}}) \,+\, D_\ell^m(r,s)\mathbf{B}_\ell^m\widehat{\mathbf{x}})\otimes\mathbf{B}_\ell^m(\widehat{\mathbf{s}})\\ &+\, E_\ell^m(r,s)\mathbf{C}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{C}_\ell^m(\widehat{\mathbf{s}})\Big)\,,\end{aligned} \tag{9.75}$$

and

$$\boldsymbol{\mathcal{G}}\,\mathbf{f} \,=\, \langle\mathbb{G}\,,\,\mathbf{f}\rangle_{(\mathcal{D}(\mathbb{R}^3)^3)',\mathcal{D}(\mathbb{R}^3)^3}\,. \tag{9.76}$$

We first clarify the notation. Here, $\mathcal{D}(\mathbb{R}^3)$ denotes the set of smooth and compactly supported functions and $\mathcal{E}(\mathbb{R}^3)$ denotes the set of smooth functions. $\mathcal{D}(\mathbb{R}^3)^3$ is the vectored-valued version and $'$ denotes the dual space, i.e. the space of functionals. For a distribution $h(r,s)\in\mathcal{E}'(\mathbb{R}_r^+\times\mathbb{R}_s^+)$ and smooth vectors $\mathbf{V}(r,\hat{\mathbf{x}}), \mathbf{W}(s,\hat{\mathbf{x}})$ defined in terms of spherical coordinates, we define the action of $h(r,s)\mathbf{V}(\hat{\mathbf{x}})\otimes\mathbf{W}(\hat{\mathbf{s}})$ on a compactly supported smooth vector-valued function $\mathbf{f}$ by,

$$\begin{aligned}&\Big\langle h(r,s)\,\mathbf{V}(r,\hat{\mathbf{x}})\otimes\mathbf{W}(s,\hat{\mathbf{s}})\,,\,\mathbf{f}\Big\rangle_{(\mathcal{E}(\mathbb{R}^3)^3)',\mathcal{E}(\mathbb{R}^3)^3}\\ &:=\,\mathbf{V}(r,\hat{\mathbf{x}})\int_0^{\pi}\int_0^{2\pi}\big\langle h(r,s)\,,\,s^2\,\mathbf{W}(s,\hat{\mathbf{s}})\cdot\mathbf{f}(\mathbf{s})\big\rangle_{\mathcal{E}'(\mathbb{R}_s^+),\mathcal{E}(\mathbb{R}_s^+)}\,\sin\theta_{\mathbf{s}}\,d\phi_{\mathbf{s}}\,d\theta_{\mathbf{s}}\,.\end{aligned} \tag{9.77}$$

Step 1 We start with the radial coefficient given in (9.65),

$$
\begin{aligned}
a_\ell^m \;&=\; \int_0^\infty G_\ell(r,s)\,\mathfrak{f}_\ell^m(s) \\
&=\; \int_0^\infty G_\ell(r,s)\frac{f_\ell^m(s)}{\gamma(s)\,\mathrm{p}_0(s)}\,ds \\
&+\; \int_0^\infty G_\ell(r,s)\left(-\frac{C_{12}(s)}{C_{22}(s)}\,\frac{g_\ell^m(s)}{\gamma(s)\,\mathrm{p}_0(s)\,\sqrt{\ell(\ell+1)}} \,-\, \frac{\ell(\ell+1)}{s}\partial_s\left(\frac{1}{C_{22}(s)}\,\frac{g_\ell^m(s)}{\gamma(s)\,\mathrm{p}_0(s)\,\sqrt{\ell(\ell+1)}}\right)\right)ds\,.
\end{aligned}
\tag{9.78}
$$

Since $\mathbf{f}$ is compactly supported, g_ℓ^m is of compact support in $[0,\infty)$. As a result of this

$$
\lim_{s\to 0}\frac{G_\ell(r,s)}{s}\,\frac{1}{C_{22}(s)}\,\frac{g_\ell^m(s)}{\gamma(s)\,\mathrm{p}_0(s)\,\sqrt{\ell(\ell+1)}} \;=\; 0\,. \tag{9.79}
$$

On the other hand, we recall the definition of C_{22} in the interior of the Sun (4.35d),

$$
C_{22}(r) \;=\; -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\ell(\ell+1)}{r^2} \qquad\Rightarrow\qquad \frac{1}{s\,C_{22}(s)} \;=\; \frac{s}{-\frac{\sigma^2}{\mathrm{c}_0^2}s^2 \,+\, \ell(\ell+1)}\,. \tag{9.80}
$$

In addition, due to its construction, G_ℓ is regular at $r=0$, we thus have

$$
\lim_{s\to 0}\frac{G_\ell(r,s)}{s}\,\frac{1}{C_{22}(s)}\,\frac{g_\ell^m(s)}{\gamma(s)\,\mathrm{p}_0(s)\,\sqrt{\ell(\ell+1)}} \;=\; 0\,. \tag{9.81}
$$

Given (9.79) and (9.81), we can perform the integration by parts in the last integral of the right-hand side of (9.78) and obtain

$$
a_\ell^m \;=\; \int_0^\infty \frac{G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)}\,f_\ell^m(s)\,ds \,+\, \int_0^\infty T_\ell(r,s)\,\frac{g_\ell^m(s)}{\sqrt{\ell(\ell+1)}}\,ds\,, \tag{9.82}
$$

where

$$
\boxed{T_\ell(r,s) \;=\; \frac{-\left(C_{12}(s) \,+\, \frac{\ell(\ell+1)}{s^2}\right)G_\ell(r,s) \,+\, \ell(\ell+1)\frac{\partial_s G_\ell(r,s)}{s}}{C_{22}(s)\,\gamma(s)\,\mathrm{p}_0(s)}\,.} \tag{9.83}
$$

Step 2 We next rewrite b_ℓ^m starting from (9.66).

First, from (9.82), we have

$$
\frac{1}{r}\frac{1}{C_{22}(r)}\partial_r a_\ell^m = \frac{1}{r}\frac{1}{C_{22}(r)}\partial_r\left(\int_0^\infty \frac{G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)}\,f_\ell^m(s)\,ds \,+\, \int_0^\infty T_\ell(r,s)\,\frac{g_\ell^m(s)}{\sqrt{\ell(\ell+1)}}\,ds\right), \tag{9.84}
$$

and

$$
\begin{aligned}
&\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)}\right)\frac{1}{r\,C_{22}(r)}a_\ell^m(r) \\
&= \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)}\right)\frac{1}{r\,C_{22}(r)}\left(\int_0^\infty \frac{G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)}\,f_\ell^m(s)\,ds \,+\, \int_0^\infty T_\ell(r,s)\,\frac{g_\ell^m(s)}{\sqrt{\ell(\ell+1)}}\,ds\right).
\end{aligned}
\tag{9.85}
$$

Thus, with

$$
\boxed{K_\ell(r,s) \;:=\; \frac{1}{r\,C_{22}(r)}\,\frac{\partial_r G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)} \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)}\right)\frac{1}{r\,C_{22}(r)}\,\frac{G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)}}\,, \tag{9.86}
$$

and

$$
\boxed{N_\ell(r,s) \;=\; \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}(r)}{\gamma(r)}\right)\frac{T_\ell(r,s)}{r\,C_{22}(r)} \,+\, \frac{\partial_r T_\ell(r,s)}{r\,C_{22}(r)}}\,, \tag{9.87}
$$

we have

$$
\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} = \int_0^\infty K_\ell(r,s) f_\ell^m(s) ds + \int_0^\infty N_\ell(r,s)\frac{g_\ell^m(s)}{\sqrt{\ell(\ell+1)}}ds + \frac{1}{C_{22}(r)\gamma(r)\mathrm{p}_0(r)}\,\frac{g_\ell^m(r)}{\sqrt{\ell(\ell+1)}}. \tag{9.88}
$$

Proposition 20. *By putting together the results (9.67), (9.82), and (9.88), the outgoing Green's tensor of (9.1) is*

$$\begin{aligned}\mathbb{G}(\mathbf{x},\mathbf{s}) \;=\; & \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{G_\ell(r,s)}{\gamma(s)\,\mathrm{p}_0(s)}\,\mathbf{P}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{P}_\ell^m(\widehat{\mathbf{s}}) \\ & +\sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} \frac{T_\ell(r,s)}{\sqrt{\ell(\ell+1)}}\,\mathbf{P}_\ell^m(r,\widehat{\mathbf{x}})\otimes\mathbf{B}_\ell^m(\widehat{\mathbf{s}}) \\ & \qquad + \sqrt{\ell(\ell+1)}\,K_\ell(r,s)\,\mathbf{B}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{P}_\ell^m(s,\widehat{\mathbf{s}}) \\ & \qquad + \left(N_\ell(r,s) \,+\, \frac{\delta(r-s)}{C_{22}(r)\,\gamma(r)\,\mathrm{p}_0(r)}\right)\mathbf{B}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{B}_\ell^m(\widehat{\mathbf{s}}) \\ & \qquad - \frac{\delta(r-s)}{-\sigma^2\rho_0 \,+\, \frac{\mathrm{p}_0'+\rho_0\Phi_0'}{r}}\,\mathbf{C}_\ell^m(\widehat{\mathbf{x}})\otimes\mathbf{C}_\ell^m(\widehat{\mathbf{s}})\,,\end{aligned} \tag{9.89}$$

with the kernels T_ℓ, K_ℓ and N_ℓ defined by (9.83), (9.86), and (9.87), and $\delta(\cdot)$ denoting the delta distribution.

10 Low-order radiation boundary conditions (RBC)

In this section, we construct radiation boundary conditions (RBC) for the vectorial ODE problem.

10.1 RBC for the conjugate radial coefficients

Since $\tilde{a}$ solves $-\partial_r^2\,\tilde{a} \,+\, V_\ell(r)\tilde{a} \;=\; 0$, we can use the same procedure as in [3, 6, 5] to obtain radiation boundary conditions of the form

$$\partial_r\tilde{a} \;=\; \mathcal{Z}\,\tilde{a}\,. \tag{10.1}$$

We recall that $\sqrt{\cdot}$ is the square root branch such that $\operatorname{Arg}\sqrt{\cdot}\in[0,\pi)$, while $(\cdot)^{1/2}$ is the principal square root branch with $\operatorname{Arg}(\cdot)^{1/2}\in(-\frac{\pi}{2},\frac{\pi}{2}]$. The branch $\sqrt{\cdot}$ ensures that $\operatorname{Im}\sqrt{\cdot}\geq 0$.

10.1.1 Nonlocal coefficient

Nonlocal coefficient By factorization of operator, we can always define the `non-local` radiation coefficient,

$$\boxed{\mathcal{Z}^\ell_{\text{nonlocal}}(r) \;:=\; \mathrm{i}\,\sqrt{-\,V_\ell(r)}\,.} \tag{10.2}$$

With this choice of square root, we have that the imaginary part of $\sqrt{-V_\ell(r)}$ is always positive. This means that with attenuation, the solutions vanish at infinity.

We can further rewrite the defining expression in (10.2) using the principle square root $(\cdot)^{1/2}$. We consider the following assumptions.

Assumption 8.

$$\operatorname{Im}(-V_\ell) \;\geq\; 0\,. \tag{10.3}$$

Assumption 9.

$$\operatorname{Im}\mathsf{k}^2 \;\geq\; 0\,. \tag{10.4}$$

Assumption 10.

$$(\operatorname{Arg}\mathsf{k}^2\,,\,\operatorname{Arg}(-V_\ell)) \;\neq\; (\pi\,,\,0)\,. \tag{10.5}$$

Under assumption Assumption 8

$$\boxed{\mathcal{Z}^\ell_{\text{nonlocal}} \;=\; \mathrm{i}\,(-V_\ell(r))^{1/2}}\,. \tag{10.6}$$

If k^2 and V_ℓ satisfy Assumption 8– Assumption 10, we can apply [5, Prop 33] to factor out k^2 of $(-V_\ell)^{1/2}$, such that,

$$\boxed{\mathcal{Z}^\ell_{\text{nonlocal}} \,=\, \mathrm{i}\,\mathsf{k}\, \left(\frac{-V_\ell(r)}{\mathsf{k}^2}\right)^{1/2}} . \tag{10.7}$$

Simplified non-local Recall that

$$-V_\ell(r) \,=\, Q(r) \,+\, \varepsilon_V(r)\,, \qquad \varepsilon_V \,=\, -V_\ell(r) \,-\, Q(r) \,=\, \frac{\ell(\ell+1)}{k_0^2}\,\mathsf{O}(r^{-3})\,, \tag{10.8}$$

where Q consists of the first three summands in the asymptotic expansion of $(-V_\ell)$

$$\begin{aligned} Q(r) \,&:=\, \mathsf{k}^2 \,+\, \frac{\alpha_{\text{ad}}}{r} \,-\, \frac{\mu_\ell^2 - \frac{1}{4}}{r^2} \\ &=\, \mathsf{k}^2 \,+\, \frac{\alpha_{\text{ad}}}{r} \,-\, \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} \,+\, 1 \,-\, \frac{\alpha^2}{k_0^2}\frac{\gamma - 1}{\gamma^2}\right)\frac{1}{r^2}\,. \end{aligned} \tag{10.9}$$

As we have observed in the discussion of Subsection 8.3, in particular in Figures 15 to 18, the addition of the term $2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}r^{-3}$ improves the approximation near $r = r_a$, i.e. using $\tilde{V}^{\text{app}}_{G,-3}$ in (8.65). In the context of RBC, we introduce the notation

$$Q^{\mathsf{G}}_\ell(r) \,:=\, \mathsf{k}^2 \,+\, \frac{\alpha_{\text{ad}}}{r} \,-\, \frac{\mu_\ell^2 - \frac{1}{4}}{r^2} \,+\, 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\frac{1}{r^3}\,. \tag{10.10}$$

At infinity, it maintains

$$-V_\ell(r) \,=\, Q^{\mathsf{G}}_\ell(r) \,+\, \varepsilon^{\mathsf{G}}_V(r)\,, \qquad \varepsilon_V \,=\, -V_\ell(r) \,-\, Q^{\mathsf{G}}_\ell(r) \,=\, \frac{\ell(\ell+1)}{k_0^2}\,\mathsf{O}(r^{-3})\,, \tag{10.11}$$

Under assumption that

$$\operatorname{Im} Q_\ell \,\geq\, 0 \quad , \quad \operatorname{Im} \mathsf{k}^2 \,\geq\, 0 \quad , \quad (\operatorname{Arg} \mathsf{k}^2\,,\, \operatorname{Arg}(Q_\ell)) \,\neq\, (\pi\,,\,0)\,. \tag{10.12}$$

we define the simplified nonlocal coefficient,

$$\boxed{\mathcal{Z}^\ell_{\text{snl}} \,=\, \mathrm{i}\,\mathsf{k}\left(\frac{Q}{\mathsf{k}^2}\right)^{1/2} \,=\, \mathrm{i}\,\mathsf{k}\,\left(1 \,+\, \frac{\alpha_{\text{ad}}}{\mathsf{k}}\,\frac{1}{\mathsf{k}\,r} \,+\, \frac{\frac{1}{4} - \mu_\ell^2}{(\mathsf{k}r)^2}\right)^{1/2}} . \tag{10.13}$$

Similarly under assumption that

$$\operatorname{Im} Q^{\mathsf{G}}_\ell \,\geq\, 0 \quad , \quad \operatorname{Im} \mathsf{k}^2 \,\geq\, 0 \quad , \quad (\operatorname{Arg} \mathsf{k}^2\,,\, \operatorname{Arg}(Q^{\mathsf{G}}_\ell)) \,\neq\, (\pi\,,\,0)\,, \tag{10.14}$$

we define the simplified gravity-enriched nonlocal coefficient ,

$$\boxed{\mathcal{Z}^\ell_{\mathrm{G}} \,=\, \mathrm{i}\,\mathsf{k}\left(\frac{Q^{\mathsf{G}}_\ell}{\mathsf{k}^2}\right)^{1/2} \,=\, \mathrm{i}\,\mathsf{k}\,\left(1 \,+\, \frac{\alpha_{\text{ad}}}{\mathsf{k}}\,\frac{1}{\mathsf{k}\,r} \,+\, \frac{\frac{1}{4} - \mu_\ell^2}{(\mathsf{k}r)^2} \,+\, 2\frac{G\,\mathfrak{m}}{\mathrm{c}_0^2}\,\frac{1}{(\mathsf{k}\,r)^2\,r}\right)^{1/2}} . \tag{10.15}$$

Remark 20. *See further discussion in the assumption that* $\operatorname{Im} Q_\ell > 0$ *see Appendix E.* △

Remark 21 (Comparison with the scalar equation)**.** *We recall the form of the conjugate ODE for the scalar equation from [5, Section 6.2] or [3] (in terms of* $\frac{\omega}{\mathrm{c}_0}$*),*

$$\partial_r^2 u \,=\, -Q_{scalar}\,u \quad , \quad \textit{with} \quad Q_{scalar} \,=\, \mathsf{k}^2 \,-\, \frac{\alpha}{r} \,-\, \frac{\ell(\ell+1)}{r^2} \tag{10.16}$$

and thus

$$[\mathcal{Z}_{scalar}]^\ell_{\text{nonlocal}} \,=\, \mathrm{i}\,\mathsf{k}\left(1 \,-\, \frac{\alpha}{r}\,\frac{1}{\mathsf{k}^2} \,-\, \frac{\ell(\ell+1)}{(r\mathsf{k})^2}\right)^{1/2}\,. \tag{10.17}$$

With $\alpha > 0$ and under assumption (2.1), $\alpha_{\mathrm{ad}} > 0$, we note that the first difference between the scalar case and the vectorial one is in the sign of the Coulomb potential. The second difference is the addition of the term $2\frac{G\mathrm{m}}{\mathrm{c}_0^2 r^3}$ as discussed in Subsection 8.3 which greatly improves the approximation near $r = r_a$.

Note however that the unknown of the scalar equation is $u = \sqrt{\rho_0}\mathrm{c}_0 \nabla\cdot\boldsymbol{\xi}$ which is closer to the Lagrangian perturbation of the pressure δ_p than to ξ_r. A proper comparison of the boundary conditions will require to derive, in this framework, the boundary condition for δ_p and then compare to the one for the scalar unknown u. △

10.1.2 Approximations of nonlocal coefficients

Coefficients in the HF family We next set out to approximation,

$$\left(\frac{-V_\ell(r)}{\mathsf{k}^2}\right)^{1/2} = \left(\frac{Q(r) + \varepsilon_Q(r)}{\mathsf{k}^2}\right)^{1/2} = \left(1 + \frac{\alpha_{\mathrm{ad}}}{\mathsf{k}}\frac{1}{\mathsf{k}\,r} + \frac{\frac{1}{4} - \mu_\ell^2}{(\mathsf{k}\,r)^2} + \frac{\varepsilon_Q(r)}{\mathsf{k}^2}\right)^{1/2}. \tag{10.18}$$

We recall that

$$\varepsilon_Q(r) = \mathsf{O}(r^{-3}), \qquad \text{is bounded in } k_0 \text{ thus } \mathsf{k}. \tag{10.19}$$

We use

$$(1 + z)^{1/2} = 1 + \frac{1}{2}z - \frac{1}{8}z^2 + \frac{1}{16}z^3 - \ldots, \qquad |z| < 1. \tag{10.20}$$

Using for the small quantity,

$$k^{-2}\,\epsilon, \qquad \text{where} \quad \epsilon = \frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4} - \mu^2}{r^2} + \varepsilon_Q(r), \tag{10.21}$$

we have,

$$\boxed{\left(\frac{-V_\ell}{\mathsf{k}^2}\right)^{1/2} = 1 + \mathsf{k}^{-2}\,\mathsf{O}(\epsilon) = 1 + \mathsf{k}^{-2}\,\mathsf{O}(r^{-1}).} \tag{10.22}$$

This leads us to introduce

$$\mathcal{Z}_{\mathtt{S\text{-}HF\text{-}0}} := \mathrm{i}\,\mathsf{k}\,. \tag{10.23}$$

Higher approximation gives

$$\begin{aligned}\left(\frac{-V_\ell}{\mathsf{k}^2}\right)^{1/2} &= 1 + \frac{1}{2}\mathsf{k}^{-2}\,\epsilon - \frac{\mathsf{k}^{-4}}{8}\epsilon^2 + \mathsf{k}^{-4}\,\mathsf{O}(\epsilon^3)\\ &= 1 + \frac{1}{2}\frac{1}{\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4} - \mu^2}{r^2} + \varepsilon_Q(r)\right) - \frac{\mathsf{k}^{-4}}{8}\left(\frac{\alpha_{\mathrm{ad}}^2}{r^2} + \mathsf{O}(r^{-3})\right) + \mathsf{k}^{-6}\,\mathsf{O}(\epsilon^3).\end{aligned} \tag{10.24}$$

Therefore, we have,

$$\boxed{\left(\frac{-V_\ell}{\mathsf{k}^2}\right)^{1/2} = 1 + \frac{1}{2}\frac{1}{\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4} - \mu^2}{r^2} - \frac{1}{8\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}^2}{r^2}\right) + \mathsf{k}^{-2}\,\mathsf{O}(r^{-3}).} \tag{10.25}$$

We define

$$\boxed{\mathcal{Z}_{\mathrm{S-HF-3}} := \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2}\frac{1}{\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4} - \mu^2}{r^2} - \frac{1}{8\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}^2}{r^2}\right)\right).} \tag{10.26}$$

Among therm in r of order 2, if assuming further that

$$\frac{1}{8\,|\mathsf{k}^2|}\alpha_{\mathrm{ad}}^2 \ll \left|\frac{1}{4} - \mu^2\right|, \tag{10.27}$$

we introduce

$$\boxed{\mathcal{Z}_{\mathrm{S-HF-2}} := \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4} - \mu^2}{r^2}\right)\right).} \tag{10.28}$$

If we assume

$$\left|\frac{\frac{1}{4}-\mu^2}{r} \;-\; \frac{1}{8\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}^2}{r}\right| \ll \alpha_{\mathrm{ad}}\,, \tag{10.29}$$

then we obtain

$$\boxed{\mathcal{Z}_{\mathrm{S-HF-1}} \;:=\; \mathrm{i}\,\mathsf{k}\left(1 \,+\, \frac{1}{2\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}}{r}\right)\,.} \tag{10.30}$$

The same argument applies to Q_ℓ^{G} to give the family `HFG` listed below.

Coefficients of the SAI family Consider as small quantity

$$\mathsf{k}^{-2}\,\epsilon\,, \qquad \text{with} \qquad \epsilon \,=\, \frac{\frac{\frac{1}{4}-\mu^2}{r^2} \,+\, \varepsilon_Q(r)}{1 \,+\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\,, \tag{10.31}$$

then

$$\begin{aligned}\left(\frac{-V_\ell}{\mathsf{k}^2}\right)^{1/2} &= \left(1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 \,+\, \frac{1}{2\mathsf{k}^2}\frac{\frac{\frac{1}{4}-\mu^2}{r^2} \,+\, \varepsilon_Q(r)}{1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}} \,-\, \frac{1}{8}\mathsf{k}^{-4}\left(\frac{\frac{\frac{1}{4}-\mu^2}{r^2} \,+\, \varepsilon_Q(r)}{1 \,+\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right)^2 \,+\, \mathsf{k}^{-6}\mathsf{O}(\varepsilon^3)\right) \\ &= \left(1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 \,+\, \frac{1}{2\mathsf{k}^2\,r^2}\frac{\frac{1}{4}-\mu^2}{1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}} \,-\, \frac{1}{8}\frac{1}{\mathsf{k}^4\,r^2}\left(\frac{\frac{1}{4}-\mu^2}{1 \,+\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right)^2 \,+\, \mathsf{k}^{-2}\mathsf{O}(r^{-3})\right)\,.\end{aligned} \tag{10.32}$$

Thus we introduce the SAI coefficients

$$\begin{aligned}\mathcal{Z}^\ell_{\texttt{S-SAI-0}} &= \mathrm{i}\,\mathsf{k}\left(1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2} \\ \mathcal{Z}^\ell_{\texttt{S-SAI-1}} &= \mathrm{i}\,\mathsf{k}\left(1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 \,+\, \frac{1}{2\mathsf{k}^2\,r^2}\frac{\frac{1}{4}-\mu^2}{1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right) \\ \mathcal{Z}^\ell_{\texttt{S-SAI-2}} &= \mathrm{i}\,\mathsf{k}\left(1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 \,+\, \frac{1}{2\mathsf{k}^2\,r^2}\frac{\frac{1}{4}-\mu^2}{1 \,-\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}} \,-\, \frac{1}{8\mathsf{k}^4\,r^2}\left(\frac{\frac{1}{4}-\mu^2}{1 \,+\, \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right)^2\right)\end{aligned} \tag{10.33}$$

Summary We have introduced the following ten approximations of the modal RBC, with k and α_{ad} defined in (9.12), and η_{ad} and μ_ℓ in (9.40).

– The approximate DtoN condition is given by Whittaker function,

$$\mathcal{Z}^\ell_{\mathrm{Whitt}} \;=\; -2\,\mathrm{i}\,\mathsf{k}\,\frac{\mathrm{W}'_{\mathrm{i}\eta_{\mathrm{ad}}\,,\,\mu_\ell}(-2\,\mathrm{i}\,\mathsf{k}\,r)}{\mathrm{W}_{\mathrm{i}\eta_{\mathrm{ad}}\,,\,\mu_\ell}(-2\,\mathrm{i}\,\mathsf{k}\,r)}\,. \tag{10.34}$$

– We have three conditions in the nonlocal family:

$$\mathcal{Z}^\ell_{\mathrm{nonlocal}} \;=\; \mathrm{i}\,\mathsf{k}\left(\frac{-V_\ell(r)}{\mathsf{k}^2}\right)^{1/2}\,, \tag{10.35a}$$

$$\mathcal{Z}^\ell_{\mathrm{snl}} \;=\; \mathrm{i}\,\mathsf{k}\left(1 \,+\, \frac{\alpha_{\mathrm{ad}}}{\mathsf{k}}\frac{1}{\mathsf{k}\,r} \,+\, \frac{\frac{1}{4}\,-\,\mu_\ell^2}{(\mathsf{k}r)^2}\right)^{1/2} \tag{10.35b}$$

$$\mathcal{Z}^\ell_{\mathsf{G}} \;=\; \mathrm{i}\,\mathsf{k}\left(1 \,+\, \frac{\alpha_{\mathrm{ad}}}{\mathsf{k}}\frac{1}{\mathsf{k}\,r} \,+\, \frac{\frac{1}{4}\,-\,\mu_\ell^2}{(\mathsf{k}r)^2} \,+\, 2\frac{G\,\mathfrak{m}}{c_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right)^{1/2}\,. \tag{10.35c}$$

– We have four conditions in the HF family:

$$\mathcal{Z}_{\mathrm{S-HF-0}} = \mathrm{i}\,\mathsf{k}\,, \tag{10.36a}$$

$$\mathcal{Z}_{\mathrm{S-HF-1}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}}{r}\right), \tag{10.36b}$$

$$\mathcal{Z}_{\mathrm{S-HF-2}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4}-\mu_\ell^2}{r^2}\right)\right), \tag{10.36c}$$

$$\mathcal{Z}_{\mathrm{S-HF-3}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4}-\mu_\ell^2}{r^2} - \frac{1}{8\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}^2}{r^2}\right)\right). \tag{10.36d}$$

– We introduce the HF family enriched with the gravity term, in the spirit of the enriched asymptotic of Section 8,

$$\mathcal{Z}_{\mathrm{S-HFG-0}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right), \tag{10.37a}$$

$$\mathcal{Z}_{\mathrm{S-HFG-1}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}}{r} + \frac{G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right), \tag{10.37b}$$

$$\mathcal{Z}_{\mathrm{S-HFG-2}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4}-\mu_\ell^2}{r^2}\right) + \frac{G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right), \tag{10.37c}$$

$$\mathcal{Z}_{\mathrm{S-HFG-3}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{1}{2\mathsf{k}^2}\left(\frac{\alpha_{\mathrm{ad}}}{r} + \frac{\frac{1}{4}-\mu_\ell^2}{r^2} - \frac{1}{8\,\mathsf{k}^2}\frac{\alpha_{\mathrm{ad}}^2}{r^2}\right) + \frac{G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right). \tag{10.37d}$$

– In the same spirit as the SAI family, we introduce

$$\mathcal{Z}^{\ell}_{\texttt{S-SAIG-0}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2} + \frac{2\,G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right)^{1/2}, \tag{10.38a}$$

$$\mathcal{Z}^{\ell}_{\texttt{S-G-0}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{2\,G\,\mathfrak{m}}{\mathsf{c}_0^2}\frac{1}{(\mathsf{k}\,r)^2\,r}\right)^{1/2}. \tag{10.38b}$$

– We have three conditions in the SAI family:

$$\mathcal{Z}^{\ell}_{\texttt{S-SAI-0}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2} \tag{10.39a}$$

$$\mathcal{Z}^{\ell}_{\texttt{S-SAI-1}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 + \frac{1}{2\mathsf{k}^2\,r^2}\frac{\frac{1}{4}-\mu_\ell^2}{1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right), \tag{10.39b}$$

$$\mathcal{Z}^{\ell}_{\texttt{S-SAI-2}} = \mathrm{i}\,\mathsf{k}\left(1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}\right)^{1/2}\left(1 + \frac{1}{2\mathsf{k}^2\,r^2}\frac{\frac{1}{4}-\mu_\ell^2}{1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}} - \frac{1}{8\mathsf{k}^4\,r^2}\left(\frac{\frac{1}{4}-\mu_\ell^2}{1 + \frac{\alpha_{\mathrm{ad}}}{r}\frac{1}{\mathsf{k}^2}}\right)^2\right). \tag{10.39c}$$

10.2 RBC for the original ODE coefficient a and b

Assuming $\tilde{a}$ satisfies the condition at $r = \mathfrak{r}$, $\partial_r\tilde{a} := \mathcal{Z}\,\tilde{a}$, we derive the corresponding conditions for a and b which are obtained from $\tilde{a}$ by relation (4.66)

$$\tilde{a}(r) := e^{-\frac{1}{2}\int \mathfrak{h}} a(r)\,, \tag{10.40}$$

and from (4.45),

$$b_\ell^m = \frac{\sqrt{\ell(\ell+1)}}{r}\frac{1}{C_{22}}\partial_r a + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{\sqrt{\ell(\ell+1)}}{rC_{22}}a\,. \tag{10.41}$$

Proposition 21. *Assuming that* $\tilde{a} = \tilde{a}_\ell^m$ *satisfies at* $r = \mathfrak{r}$ *condition*

$$\partial_r \tilde{a}(\mathfrak{r}) \,=\, \mathcal{Z}(\mathfrak{r})\,\tilde{a}(\mathfrak{r})\,. \tag{10.42}$$

then $a = a_\ell^m$ *and* $b = b_\ell^m$ *defined by* (10.40) *and* (10.41) *satisfy at* $r = \mathfrak{r}$,

$$\partial_r\, a_\ell^m(\mathfrak{r}) \,=\, \left(\frac{\mathfrak{h}(\mathfrak{r})}{2} \,+\, \mathcal{Z}(\mathfrak{r})\right) a_\ell^m(\mathfrak{r})\,; \tag{10.43a}$$

$$\begin{aligned}\partial_r\, \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}} \,=\, &\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} \,+\, \mathcal{Z}(\mathfrak{r}) - \frac{1}{\mathfrak{r}} \,-\, \frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right) \frac{b(\mathfrak{r}}{\sqrt{\ell(\ell+1)}} \\ &+\, \left(\frac{\mathfrak{h}^2(\mathfrak{r})}{4} \,-\, \mathcal{Z}^2(\mathfrak{r}) \,-\, \frac{2}{\mathfrak{r}^2} \,+\, \mathfrak{g}(\mathfrak{r})\right) \frac{1}{\mathfrak{r}\, C_{22}(\mathfrak{r})} a(\mathfrak{r})\,. \end{aligned}\tag{10.43b}$$

Proof. Since

$$a \,=\, e^{\int \frac{1}{2}\mathfrak{h}}\,\tilde{a}\,, \text{and} \quad \partial_r(e^{\int \frac{1}{2}h}\,\tilde{a}) \,=\, e^{\int \frac{1}{2}h}\left(\partial_r\tilde{a} \,+\, \frac{1}{2}\mathfrak{h}\,\tilde{a}\right) \,=\, e^{\frac{1}{2}h}\partial_r\tilde{a} \,+\, \frac{1}{2}\mathfrak{h}\,a\,,$$

we have,

$$\partial_r a \,=\, e^{\int \frac{1}{2}h}\partial_r\tilde{a} \,+\, \frac{1}{2}\mathfrak{h}\,a\,. \tag{10.44}$$

Evaluating at $r = \mathfrak{r}$, it gives

$$\partial_r a\,(\mathfrak{r}) \,=\, e^{\int \frac{1}{2}h}\Big|_{\mathfrak{r}}\,\partial_r\tilde{a}(\mathfrak{r}) \,+\, \tfrac{1}{2}\mathfrak{h}(\mathfrak{r})\,a\,(\mathfrak{r}) \,=\, e^{\int \frac{1}{2}h}\Big|_{\mathfrak{r}}\,\mathcal{Z}(\mathfrak{r})\,\tilde{a}(\mathfrak{r}) \,+\, \frac{1}{2}\mathfrak{h}(\mathfrak{r})\,a\,(\mathfrak{r}) \,=\, \mathcal{Z}(\mathfrak{r})\,a(\mathfrak{r}) \,+\, \frac{1}{2}\mathfrak{h}(\mathfrak{r})\,a\,(\mathfrak{r})\,.$$

We thus obtain the boundary condition (10.43a) for a.

Since a is a solution for an ODE of order 2, this also gives the values of $\partial_r^2 a$. Recall from (4.62) that

$$\partial_r^2 a \,=\, \mathfrak{h}(r)\,\partial_r a \,+\, \mathfrak{g}(r)\,a\,. \tag{10.45}$$

Evaluating both sides at $r = \mathfrak{r}$ and replacing $\partial_r a$ using (10.43a) gives

$$\partial_r^2 a(\mathfrak{r}) \,=\, \mathfrak{h}(\mathfrak{r})\,\partial_r a(\mathfrak{r}) \,+\, \mathfrak{g}(\mathfrak{r})\,a(\mathfrak{r}) \,\overset{(10.43a)}{=}\, \mathfrak{h}(\mathfrak{r})\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} \,+\, \mathcal{Z}(\mathfrak{r})\right) a(\mathfrak{r}) \,+\, \mathfrak{g}(\mathfrak{r})\,a(\mathfrak{r})\,, \tag{10.46}$$

thus

$$\partial_r^2 a(\mathfrak{r}) \,=\, \left(\frac{\mathfrak{h}^2(\mathfrak{r})}{2} \,+\, \mathfrak{h}(\mathfrak{r})\,\mathcal{Z}(\mathfrak{r}) \,+\, \mathfrak{g}(\mathfrak{r})\right) a(\mathfrak{r})\,. \tag{10.47}$$

We next consider the RBC for

$$\tilde{b}_\ell^m \,:=\, \frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\,. \tag{10.48}$$

From the definition of b_ℓ^m in (4.45), we have

$$\tilde{b}_\ell^m \,=\, \frac{1}{r\,C_{22}}\left(\partial_r a \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha}{\gamma}\right) a\right)\,. \tag{10.49}$$

Evaluating at $r = \mathfrak{r}$, this gives

$$\tilde{b}_\ell^m(\mathfrak{r}) \,=\, \frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\left(\partial_r a(\mathfrak{r}) \,+\, \left(\frac{2}{\mathfrak{r}} \,-\, \frac{\alpha}{\gamma}\right) a(\mathfrak{r})\right) \,=\, \frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} \,+\, \mathcal{Z}(\mathfrak{r}) \,+\, \frac{2}{\mathfrak{r}} \,-\, \frac{\alpha}{\gamma}\right) a(\mathfrak{r})\,. \tag{10.50}$$

On the other hand, if we differentiate both sides of (10.49), we obtain

$$\begin{aligned}\partial_r\,\tilde{b} \,&=\, \left(\frac{1}{r}\frac{1}{C_{22}}\partial_r a \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}a\right)' \\ &=\, \frac{1}{r}\frac{1}{C_{22}}\partial_r^2 a \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\partial_r a + \left(\frac{1}{r}\frac{1}{C_{22}}\right)'\partial_r a \,+\, \left(\left(\frac{2}{r} \,-\, \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\right)' a\,.\end{aligned}$$

Rewriting the last three terms on the rhs by expanding the last derivative, we obtain,

$$\partial_r \tilde{b} = \frac{1}{r\,C_{22}}\partial_r^2 a + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\partial_r a \tag{10.51}$$

$$+ \left(\frac{1}{r\,C_{22}}\right)'\left(\partial_r a + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right) a\right) + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)'\frac{1}{rC_{22}} a\,. \tag{10.52}$$

Evaluating at $r = \mathfrak{r}$ and using (10.43a) to replace $\partial_r a(\mathfrak{r})$ and (10.47) $\partial_r^2 a(\mathfrak{r})$, then we have,

$$\begin{aligned}
&\frac{1}{r\,C_{22}}\partial_r^2 a + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)\frac{1}{rC_{22}}\partial_r a \\
&\overset{r=\mathfrak{r}}{=} \frac{\mathfrak{h}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r})\right) a(\mathfrak{r}) + \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) \\
&\qquad + \left(\frac{2}{\mathfrak{r}} - \frac{\alpha}{\gamma}\right)\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r})\right) a_\ell^m(\mathfrak{r}) \\
&= \frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r})\right)\left(\mathfrak{h}(\mathfrak{r}) + \frac{2}{\mathfrak{r}} - \frac{\alpha}{\gamma}\right) a(\mathfrak{r}) + \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) \\
&= \left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r})\right)\left(\tilde{b}(\mathfrak{r}) - \frac{\mathcal{Z}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) + \frac{\mathfrak{h}(\mathfrak{r})}{2\mathfrak{r}C_{22}(\mathfrak{r})} a(\mathfrak{r})\right) + \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r})\,,
\end{aligned}$$

and the second line of the right-hand side of (10.51) is written as

$$\begin{aligned}
&\left(\frac{1}{r\,C_{22}}\right)'\left(\partial_r a + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right) a\right) + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)'\frac{1}{rC_{22}} a \\
&\overset{r=\mathfrak{r}}{=} \left(\frac{1}{r\,C_{22}}\right)'\Big|_{r=\mathfrak{r}}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r}) + \left(\frac{2}{\mathfrak{r}} - \frac{\alpha}{\gamma}\right)\right) a(\mathfrak{r}) + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)'\Big|_{r=\mathfrak{r}}\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})} a(\mathfrak{r}) \\
&= \left(\frac{1}{r\,C_{22}}\right)'\Big|_{r=\mathfrak{r}}\mathfrak{r}\,C_{22}(\mathfrak{r})\,\tilde{b}(\mathfrak{r}) + \left(\frac{2}{r} - \frac{\alpha}{\gamma}\right)'\Big|_{r=\mathfrak{r}}\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})} a(\mathfrak{r}) \\
&= -\left(\frac{1}{\mathfrak{r}} + \frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r}) - \frac{2}{\mathfrak{r}^3\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r})\,.
\end{aligned}$$

Combining these above calculations, we obtain a boundary condition for $\tilde{b}$

$$\begin{aligned}
\partial_r \tilde{b}(\mathfrak{r}) &= \left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r})\right)\left(\tilde{b}(\mathfrak{r}) - \frac{\mathcal{Z}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) + \frac{\mathfrak{h}(\mathfrak{r})}{2\,\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r})\right) + \frac{\mathfrak{g}(\mathfrak{r})}{\mathfrak{r}\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) \\
&\qquad - \left(\frac{1}{\mathfrak{r}} + \frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r}) - \frac{2}{\mathfrak{r}^3\,C_{22}(\mathfrak{r})}\, a(\mathfrak{r}) \\
&= \left(\frac{\mathfrak{h}(\mathfrak{r})}{2} + \mathcal{Z}(\mathfrak{r}) - \frac{1}{\mathfrak{r}} - \frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r}) \\
&\qquad + \left(\frac{\mathfrak{h}^2(\mathfrak{r})}{4} - \mathcal{Z}^2(\mathfrak{r}) - \frac{2}{\mathfrak{r}^2} + \mathfrak{g}(\mathfrak{r})\right)\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})} a(\mathfrak{r})\,.
\end{aligned} \tag{10.53}$$

□

10.3 Radiation conditions in 3D

In this subsection, we give a preliminary discussion of 3D RBC to be used in a 3D discretization of (3.1). Since a choice of 3D RBC depends on the variational formulation of (3.1) and on the type of discretization method (e.g. Discontinuous Galerkin, Finite Elements), we only give some variants that come directly from the modal conditions (3.1). Specifically, in (10.43), if we assume that $\mathcal{Z}$ is independent of ℓ (e.g., $\mathcal{Z}_{\texttt{S-HF-0}}$ and $\mathcal{Z}_{\texttt{S-HF-1}}$ in (10.36)), and using the approximate version of $\mathfrak{h}_\ell$ that is independent of ℓ, we obtain readily a boundary condition for $\boldsymbol{\xi}$ placed on the boundary of a sphere. As one of our conditions, (10.81) and (10.84), which bear some resemblance to a common noneffective boundary condition (10.85) (or (10.87)) employed in [18, 7, 34, 29]. However, the crucial difference is in the leading wavenumber, as discussed in Subsection 10.3.3.

We first recall the decomposition of $\boldsymbol{\xi}$,

$$\boldsymbol{\xi} \,=\, \xi_r \mathbf{e}_r \,+\, \boldsymbol{\xi}_h\,, \qquad \xi_r \,=\, \boldsymbol{\xi}\,\cdot\,\mathbf{e}_r \,=\, \pi_r\,\boldsymbol{\xi}\,, \qquad \boldsymbol{\xi}_h \,=\, \boldsymbol{\xi}\,-\,\boldsymbol{\xi}_r\,, \tag{10.54}$$

and

$$\xi_r \mathbf{e}_r \,=\, \mathbf{e}\otimes\mathbf{e}_r\cdot\boldsymbol{\xi}\,, \qquad \boldsymbol{\xi}_h \,=\, (\mathbf{Id}\,-\,\mathbf{e}\otimes\mathbf{e}_r)\cdot\boldsymbol{\xi}_h\,. \tag{10.55}$$

For simplicity, we assume that the source has zero $\mathbb{C}_\ell^m$ component, see (4.5). From the results in (4.23)–(4.26), the tangential part of $\boldsymbol{\xi}$ only has component along $\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m$,

$$\begin{aligned}
\xi_r \,&=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} a_\ell^m(r)\,\mathrm{Y}_\ell^m\,;\\
\boldsymbol{\xi}_h \,&=\, \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{b_\ell^m(r)}{\sqrt{\ell(\ell+1)}}\nabla_{\mathbb{S}^2}\,\mathrm{Y}_\ell^m\,.
\end{aligned}$$

10.3.1 First variant

We first deduce the boundary condition for $\boldsymbol{\xi}$ in series form.

Lemma 4. *Assuming that a_ℓ^m satisfies the condition (10.43a) and b_ℓ^m (10.43b),*

$$\begin{aligned}
\nabla\boldsymbol{\xi}\cdot\mathbf{e}_r \,=\, &\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2}+\mathcal{Z}(\mathfrak{r})\right)\left(a_\ell^m(\mathfrak{r})\,\mathrm{Y}_\ell^m\,\mathbf{e}_r+\tilde{b}(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\right)\\
&-\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{1}{\mathfrak{r}}\,+\,\frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{\mathfrak{h}^2(\mathfrak{r})}{4}\,-\,\mathcal{Z}^2(\mathfrak{r})\,-\,\frac{2}{\mathfrak{r}^2}\,+\,\mathfrak{g}(\mathfrak{r})\right)\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}a(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,.
\end{aligned}\tag{10.56}$$

Proof. From the expression of the gradient of a vector in spherical basis(2.13), we have

$$\nabla\boldsymbol{\xi}\cdot\mathbf{e}_r = \partial_r(\pi_r\boldsymbol{\xi})\,\mathbf{e}_r\,+\,\partial_r(\pi_\theta\boldsymbol{\xi})\,\mathbf{e}_\theta\,+\,\partial_r(\pi_\phi\boldsymbol{\xi})\,\mathbf{e}_\phi\,, \tag{10.57}$$

thus

$$\begin{aligned}
\nabla\boldsymbol{\xi}\cdot\mathbf{e}_r \,&=\, \partial_r\xi_r\mathbf{e}_r\,+\,\partial_r\boldsymbol{\xi}_h\\
&=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\partial_r a_\ell^m)\,\mathrm{Y}_\ell^m\,\mathbf{e}_r\,+\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\partial_r\frac{b_\ell^m}{\sqrt{\ell(\ell+1)}}\right)\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,.
\end{aligned}\tag{10.58}$$

We next use (10.43) to replace $\partial_r a_\ell^m$ and $\partial_r b_\ell^m$,

$$\begin{aligned}
\partial_r a &= \left(\frac{\mathfrak{h}}{2}+\mathcal{Z}\right)a\,,\\
\partial_r\tilde{b} &= \left(\frac{\mathfrak{h}(\mathfrak{r})}{2}\,+\,\mathcal{Z}(\mathfrak{r})-\frac{1}{\mathfrak{r}}\,-\,\frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r})+\left(\frac{\mathfrak{h}^2(\mathfrak{r})}{4}\,-\,\mathcal{Z}^2(\mathfrak{r})\,-\,\frac{2}{\mathfrak{r}^2}\,+\,\mathfrak{g}(\mathfrak{r})\right)\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}a(\mathfrak{r})\,,
\end{aligned}$$

to obtain

$$\begin{aligned}
\nabla\boldsymbol{\xi}\cdot\mathbf{e}_r \,=\, &\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2}+\mathcal{Z}(\mathfrak{r})\right)a_\ell^m(\mathfrak{r})\,\mathrm{Y}_\ell^m\,\mathbf{e}_r\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{\mathfrak{h}(\mathfrak{r})}{2}\,+\,\mathcal{Z}(\mathfrak{r})-\frac{1}{\mathfrak{r}}\,-\,\frac{C_{22}'(\mathfrak{r})}{C_{22}(\mathfrak{r})}\right)\tilde{b}(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\\
&+\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{\mathfrak{h}^2(\mathfrak{r})}{4}\,-\,\mathcal{Z}^2(\mathfrak{r})\,-\,\frac{2}{\mathfrak{r}^2}\,+\,\mathfrak{g}(\mathfrak{r})\right)\frac{1}{\mathfrak{r}\,C_{22}(\mathfrak{r})}a(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,.
\end{aligned}\tag{10.59}$$

□

An approximation of (10.56) is obtained using the approximations of $\mathcal{Z}$, $\mathfrak{h}$ and $\mathfrak{g}$ that are independent of ℓ. From the derivation in Subsection 10.1 and Proposition 18, this means that we work modulo r^{-2}. Recall from Proposition 17,

$$\frac{C_{22}'}{C_{22}} = k_0^{-2} \underbrace{\mathsf{O}(r^{-2})}_{\text{bounded with respect to } k_0} . \tag{10.60}$$

From the result of Proposition 18, we have

$$\mathfrak{h}(r) = \alpha - \frac{2}{r} + \mathsf{O}(r^{-2}) , \qquad \mathfrak{g} = -k_0^2 + 2\left(\alpha - \frac{\alpha}{\gamma}\right)\frac{1}{r} + \mathsf{O}(r^{-2}) . \tag{10.61}$$

We consider $\mathcal{Z}$ independent of m and ℓ and of the form

$$\mathcal{Z}(r) = \mathcal{Z}_0 + \frac{\mathcal{Z}_1}{r} . \tag{10.62}$$

Then

$$\mathfrak{h}^2 = \alpha^2 - \frac{4\,\alpha}{r} + \mathsf{O}(r^{-2}) ; \tag{10.63}$$

$$\mathcal{Z}^2(r) = \mathcal{Z}_0^2 + \frac{2\,\mathcal{Z}_0\,\mathcal{Z}_1}{r} + \mathsf{O}(r^{-2}) ; \tag{10.64}$$

$$\frac{1}{r\,C_{22}} = -\frac{1}{k_0^2}\frac{1}{r} + k_0^{-2}\mathsf{O}(r^{-2}) . \tag{10.65}$$

This leads to

$$\begin{aligned} &\frac{\mathfrak{h}^2(r)}{4} - \mathcal{Z}^2(r) - \frac{2}{r^2} + \mathfrak{g}(r) \\ &= \frac{\alpha^2}{4} - \frac{\alpha}{r} - \mathcal{Z}_0^2 - \frac{2\,\mathcal{Z}_0\,\mathcal{Z}_1}{r} - k_0^2 + 2\left(\alpha - \frac{\alpha}{\gamma}\right)\frac{1}{r} + \mathsf{O}(r^{-2}) , \end{aligned} \tag{10.66}$$

and

$$\left(\frac{\mathfrak{h}^2(r)}{4} - \mathcal{Z}^2(r) - \frac{2}{r^2} + \mathfrak{g}(r)\right)\frac{1}{r\,C_{22}} = \frac{\frac{\alpha^2}{4} - \mathcal{Z}_0^2 - k_0^2}{k_0^2\,r} + \mathsf{O}(r^{-2}) . \tag{10.67}$$

Using these approximations, we derive

$$\begin{aligned} \nabla\boldsymbol{\xi}\cdot\mathbf{e}_r \sim \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}&\left(\frac{\alpha}{2} - \frac{1}{\mathfrak{r}} + \mathcal{Z}(\mathfrak{r})\right)\left(a_\ell^m(\mathfrak{r})\,\mathrm{Y}_\ell^m\,\mathbf{e}_r + \tilde{b}_\ell^m\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\right) \\ &- \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{\mathfrak{r}}\tilde{b}(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m \\ &+ \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\frac{\alpha^2}{4} - \mathcal{Z}_0^2 - k_0^2}{k_0^2\,\mathfrak{r}}\,a_\ell^m(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m . \end{aligned} \tag{10.68}$$

Finally, using the property (2.15b) of Y_ℓ^m,

$$\sum_{m=-\ell}^{\ell} a(\mathfrak{r})\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m = \nabla_{\mathbb{S}^2}\sum_{m=-\ell}^{\ell} a(\mathfrak{r})\,\mathrm{Y}_\ell^m = \nabla_{\mathbb{S}^2}\,\mathbf{e}_r\cdot\boldsymbol{\xi} , \tag{10.69}$$

we arrive at the following approximate condition of (10.56), with $\mathcal{Z}$ independent of ℓ,

$$\boxed{\nabla\boldsymbol{\xi}\cdot\mathbf{e}_r = \left(\frac{\alpha}{2} - \frac{1}{\mathfrak{r}} + \mathcal{Z}(\mathfrak{r}) - \frac{\mathbb{I} - \mathbf{e}_r\otimes\mathbf{e}_r}{\mathfrak{r}} + \frac{\frac{\alpha^2}{4} - \mathcal{Z}_0^2 - k_0^2}{k_0^2\,\mathfrak{r}}\nabla_{\mathbb{S}^2}(\mathbf{e}_r\,\cdot)\right)\boldsymbol{\xi} . \qquad (10.70)}$$

10.3.2 Second variant

With the same derivation, we obtain the following lemma.

Lemma 5. *Assuming that a_ℓ^m satisfies condition (10.43a), then at $r = \mathfrak{r}$,*

$$\begin{aligned}\boldsymbol{\tau}\cdot\mathbf{e}_r \,=\, &\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left(\gamma\,\mathrm{p}_0\left(\frac{\mathfrak{h}}{2}\,+\,\mathcal{Z}\right)\,+\,\frac{2}{r}(\gamma-1)\mathrm{p}_0\right)a_\ell^m\,\mathrm{Y}_\ell^m\,\mathbf{e}_r\\ &+\,\frac{(\gamma-1)\,\mathrm{p}_0}{r}(\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h)\,\mathbf{e}_r\,+\,\frac{\mathrm{p}_0}{r}\,\nabla_{\mathbb{S}^2}\,(\boldsymbol{\xi}\cdot\mathbf{e}_r)\,-\,\frac{\mathrm{p}_0}{r}\boldsymbol{\xi}_h\,,\qquad |\mathbf{x}|\,=\,\mathfrak{r}\,.\end{aligned}\tag{10.71}$$

Proof. We recall definition of second-order tensor $\boldsymbol{\tau}$ defined in (3.22),

$$\begin{aligned}\boldsymbol{\tau}\,:=\,(\gamma-1)\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\,\mathbf{Id}\,+\,\mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\\ \Rightarrow\quad \boldsymbol{\tau}\cdot\mathbf{e}_r\,=\,(\gamma-1)\,\mathrm{p}_0\,(\nabla\cdot\boldsymbol{\xi})\,\mathbf{e}_r\,+\,\mathrm{p}_0\,(\nabla^t\boldsymbol{\xi})\cdot\mathbf{e}_r\,.\end{aligned}\tag{10.72}$$

We recall from (4.4) that

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi}\,&=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\frac{(r^2a_\ell^m)'}{r^2}\,-\,\sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\mathrm{Y}_\ell^m\\ &=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\partial_r a_\ell^m\,+\,\frac{2}{r}a_\ell^m\,-\,\sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\mathrm{Y}_\ell^m\,.\end{aligned}\tag{10.73}$$

This can also be written as

$$\begin{aligned}\nabla\cdot\boldsymbol{\xi}\,&=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\frac{(r^2a_\ell^m)'}{r^2}\,-\,\sqrt{\ell(\ell+1)}\,\frac{b_\ell^m}{r}\right]\mathrm{Y}_\ell^m\\ &=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\partial_r a_\ell^m\,+\,\frac{2}{r}a_\ell^m\right]\mathrm{Y}_\ell^m\,+\,\frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\,.\end{aligned}\tag{10.74}$$

Thus it remains to examine the second term $(\nabla^t\boldsymbol{\xi})\cdot\mathbf{e}_r$. From the expression (2.13) of $\nabla\boldsymbol{\xi}$ in spherical coordinates, and noting that $\xi_r = \pi_r\boldsymbol{\xi}$, we have

$$\begin{aligned}(\nabla\boldsymbol{\xi})^t\,\cdot\,\mathbf{e}_r\,&=\,(\partial_r\xi_r)\,\mathbf{e}_r\,+\,\left(\frac{\partial_\theta\xi_r}{r}\,-\,\frac{\pi_\theta\boldsymbol{\xi}}{r}\right)\mathbf{e}_\theta\,+\,\left(\frac{\partial_\theta\xi_r}{r\sin\theta}-\frac{\pi_\phi\boldsymbol{\xi}}{r}\right)\mathbf{e}_\phi\\ &=\,(\partial_r\xi_r)\,\mathbf{e}_r\,+\,\frac{1}{r}\underbrace{\left(\partial_\theta\,\xi_r\,\mathbf{e}_\theta\,+\,\frac{\partial_\theta\,\xi_r}{r\sin\theta}\,\mathbf{e}_\phi\right)}_{\nabla_{\mathbb{S}^2}\,\xi_r}\,-\,\frac{1}{r}\underbrace{\Big((\pi_\theta\boldsymbol{\xi})\,\mathbf{e}_\theta\,+\,(\pi_\phi\boldsymbol{\xi})\,\mathbf{e}_\phi\Big)}_{\boldsymbol{\xi}_h}\,.\end{aligned}$$

Since

$$\nabla_{\mathbb{S}^2}\xi_r\,=\,\nabla_{\mathbb{S}^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a_\ell^m(r)\mathrm{Y}_\ell^m\,=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}a_\ell^m(r)\,\nabla_{\mathbb{S}^2}\mathrm{Y}_\ell^m\,,$$

we have

$$(\nabla\boldsymbol{\xi})^t\,\cdot\,\mathbf{e}_r\,=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\partial_r\,a_\ell^m)\,\mathrm{Y}_\ell^m\,\mathbf{e}_r\,+\,\frac{1}{r}\nabla_{\mathbb{S}^2}(\boldsymbol{\xi}\cdot\mathbf{e}_r)\,-\,\frac{1}{r}\boldsymbol{\xi}_h\,.\tag{10.75}$$

Using (10.75) together with (10.74), we obtain

$$\begin{aligned}\boldsymbol{\tau}\cdot\mathbf{e}_r\,:=\,&(\gamma-1)\,\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\,\mathbf{e}_r\,+\,\mathrm{p}_0\,\nabla^t\boldsymbol{\xi}\cdot\mathbf{e}_r\\ =\,&(\gamma-1)\,\mathrm{p}_0\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\partial_r a_\ell^m\,+\,\frac{2}{r}a_\ell^m\right]\mathrm{Y}_\ell^m\,\mathbf{e}_r\,+\,\frac{(\gamma-1)\,\mathrm{p}_0}{r}(\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h)\,\mathbf{e}_r\\ &+\,\mathrm{p}_0\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(\partial_r\,a_\ell^m)\,\mathrm{Y}_\ell^m\,\mathbf{e}_r\,+\,\frac{\mathrm{p}_0}{r}\,\nabla_{\mathbb{S}^2}\,(\boldsymbol{\xi}\cdot\mathbf{e}_r)\,-\,\frac{\mathrm{p}_0}{r}\boldsymbol{\xi}_h\,.\end{aligned}\tag{10.76}$$

The derivation is finished by using (10.43a) to replace $\partial_r a_\ell^m$.

□

As in Subsection 10.3.1, using an approximation of $\mathcal{Z}$ and $\mathfrak{h}$ that are independent of m and ℓ, we derive the following approximate version of (10.71),

$$\boxed{\begin{aligned}\boldsymbol{\tau}\cdot\mathbf{e}_r \,=\, & \left(\gamma\,\mathrm{p}_0\left(\frac{\alpha}{2}+\frac{1}{r}+\mathcal{Z}\right)+\frac{2}{r}(\gamma-1)\mathrm{p}_0\right)\mathbf{e}_r\otimes\mathbf{e}_r\;\boldsymbol{\xi} \\ & +\frac{(\gamma-1)\,\mathrm{p}_0}{r}(\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h)\,\mathbf{e}_r+\frac{\mathrm{p}_0}{r}\nabla_{\mathbb{S}^2}(\boldsymbol{\xi}\cdot\mathbf{e}_r)-\frac{\mathrm{p}_0}{r}\boldsymbol{\xi}_h\,.\end{aligned}} \tag{10.77}$$

We can obtain a lower order condition by ignoring all terms containing the factor $\frac{\mathrm{p}_0}{r}$ in (10.77). In particular, from (10.76), we have

$$\boldsymbol{\tau}\cdot\mathbf{e}_r \,=\, \gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}+\frac{\mathrm{p}_0}{r}(\ldots) \tag{10.78}$$

such that the right-hand side of (10.77) is approximated by $\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}$. With k defined in (9.12), for $\mathcal{Z}$ of the form

$$\mathcal{Z}\,=\,\mathrm{i}\,\mathsf{k}+\mathrm{i}\frac{\mathcal{Z}_{-1}}{r}\,, \tag{10.79}$$

if on both sides of (10.77), we ignore the term containing factor $\frac{\mathrm{p}_0}{r}$, we arrive at the following approximate condition,

$$\gamma\mathrm{p}_0\nabla\cdot\boldsymbol{\xi}\,=\,\gamma\mathrm{p}_0\left(\frac{\alpha}{2}+\mathrm{i}\mathsf{k}\right)\boldsymbol{\xi}\cdot\mathbf{e}_r\,, \tag{10.80}$$

which simplifies to

$$\boxed{\nabla\cdot\boldsymbol{\xi}\,=\,\left(\frac{\alpha}{2}+\mathrm{i}\,\mathsf{k}\right)\boldsymbol{\xi}\cdot\mathbf{e}_r\,.} \tag{10.81}$$

This one resembles the form of non-reflective boundary condition used in literature, see discussion below.

10.3.3 Third variant

From (10.74), we have,

$$\nabla\cdot\boldsymbol{\xi}\,=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left[\partial_r a_\ell^m+\frac{2}{r}a_\ell^m\right]\mathrm{Y}_\ell^m+\frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\,. \tag{10.82}$$

The following statement follows readily using (10.43a) to replace $\partial_r a_\ell^m$.

Lemma 6. *Assuming that a_ℓ^m satisfies condition (10.43a), then*

$$\nabla\cdot\boldsymbol{\xi}\,=\,\sum_{\ell=0}^{\infty}\sum_{m=-\ell}\left(\frac{\mathfrak{h}}{2}+\mathcal{Z}+\frac{2}{r}\right)a_\ell^m\,\mathrm{Y}_\ell^m+\frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\,. \tag{10.83}$$

As before, using approximations of $\mathcal{Z}$ and $\mathfrak{h}$ that are independent of m and ℓ, we obtain,

$$\boxed{\nabla\cdot\boldsymbol{\xi}\,=\,\left(\frac{\alpha}{2}+\frac{1}{r}+\mathcal{Z}\right)\boldsymbol{\xi}\cdot\mathbf{e}_r+\frac{1}{r}\nabla_{\mathbb{S}^2}\cdot\boldsymbol{\xi}_h\,.} \tag{10.84}$$

The condition (10.84) and in particular (10.81) are similar to the nonreflective boundary condition employed in [18, 7, 34] and [29] in the absence of flow. Based on the work of [18, section 3.3 p. 89], the condition in the absence of flow (i.e. $\mathbf{v}_0=0$) simplifies to an impedance condition in the form of a ratio between the Lagrangian pressure perturbation $\delta_\mathrm{p}^\mathrm{L}$ and the normal direction of displacement $\boldsymbol{\xi}$,

$$\delta_\mathrm{p}^\mathrm{L}\,=\,-\mathrm{i}\,\omega\,\rho_0\,\mathrm{c}_0\,\boldsymbol{\xi}\cdot\mathbf{n}\,. \tag{10.85}$$

This is implemented in [7, Eqn 5.28] (in particular with the formulation of Galbrun's Eqn 5.9 and non-reflective condition Eqn 5.18 which is the same as Eqn 5.28 for no flow), and [34, Eqn 2.14], [33, Eqn 4.1] following[29, Eqn 4.3]. Since

$$\delta_\mathrm{p}^\mathrm{L}\,=\,-\rho_0\,\mathrm{c}_0^2\nabla\cdot\boldsymbol{\xi}\,, \tag{10.86}$$

the condition (10.85) is equivalent to

$$\nabla\cdot\boldsymbol{\xi}\,=\,\mathrm{i}\frac{\omega}{\mathrm{c}_0}\,\boldsymbol{\xi}\cdot\mathbf{n}\,. \tag{10.87}$$

However, in the aforementioned references, instead of k defined in (9.12) in (10.81), the wavenumber $\frac{\omega}{\mathrm{c}_0}$ is used. This is the most important difference with our result.

11 Conclusions

We consider the propagation of time-harmonic waves in the Sun in the vectorial form, and we have achieved the following results:

1. Starting from the Galbrun equation, under S+AtmoCAI assumption, we obtain the coupled system whose unknowns are the radial and tangential coefficients of $\boldsymbol{\xi}$ in vector spherical harmonic basis, denoted by a_ℓ^m and b_ℓ^m. We then obtain a decoupled problem solved only by a_ℓ^m, also called the modal radial ODE. With no-source, this is written as

$$\big(\hat{q}\partial_r^2 \; + \; q\,\partial_r \; + \; \tilde{q}\big)a_\ell^m \; = \; 0\,, \tag{11.1}$$

   or equivalently

$$(\partial_r^2 \; + \; V)\tilde{a}_\ell^m \; = \; 0\,. \tag{11.2}$$

   Explicit and compact expressions for $\hat{q}$, q, $\tilde{q}$ as well as V are given. For the interior, we also identify our derivation with the coupled system given in [35] and [11]. A second derivation for the radial ODE for a_ℓ^m is given starting from the decoupled system of [35] and [11].

2. We give complete indicial analysis for the above ODEs, i.e. for both the interior and in the atmosphere, with and without attenuation. We note that the set of real singularities differs with or without attenuation. With less restrictive hypotheses on the background coefficients, we obtain the same result at $r = 0$ as in [35] for $\ell > 0$, with the indicial exponents being $\ell - 1$ and $-\ell - 2$. For $\ell = 0$, our analysis shows that the exponents are 1 and -2.

3. We obtain asymptotic description for V using two different approaches. This allows to define outgoing solutions and obtain a characterization of such solutions in terms of an oscillatory phase.

4. The indicial analysis and asymptotic analysis at infinity are put together to construct outgoing solutions and outgoing Green kernel globally (i.e on $(0, \infty)$). From the modal Green kernel, one can obtain the 3D Green kernel.

5. We have also obtained low-order radiation boundary conditions both in modal form (i.e., for the coefficients of the decomposition in vector spherical harmonics) and in 3D form. These boundary conditions are now for the 3D unknown $\boldsymbol{\xi}$ and extend the results from [5, 6, 4] that have been derived for an unknown linked to $\nabla \cdot \boldsymbol{\xi}$ which is related to the Lagrangian perturbation of the pressure.

This work will allow to extend the framework developed in [20] for a scalar wave equation to a vectorial problem. The Green kernel derived here is the main input for the computation of Born sensitivity kernels that relate the helioseismic observables to the perturbations in the solar interior. This improvement in the forward modeling should lead to a better understanding of the data and correction of the systematic errors. Moreover, the newly derived boundary conditions are necessary to solve numerically the vectorial equation. They can also be used in the presence of flows in the solar interior and some new physical problems that could not be studied with the scalar equation can now be treated. Some important applications are the analysis of the recently discovered Rossby waves [25] or the internal gravity modes [16].

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A Explicit expressions of the coefficients of the radial ODE in the interior

In this appendix, we prove Proposition 4, showing (4.53) and (4.55). We first recall from (4.35), such that

$$C_{11} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,+\, 2\frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,+\, \frac{2}{r^2} \,-\, 2\frac{\alpha_{\mathrm{p}_0}}{r\gamma}\,; \tag{A.1a}$$

$$C_{12} \,=\, \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{r\,\gamma} \,-\, \frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,-\, \frac{1}{r^2}\right)\,; \tag{A.1b}$$

$$C_{22} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\ell(\ell+1)}{r^2}\,. \tag{A.1c}$$

$$\hat{q}(r) \,=\, -1 \,+\, \frac{\ell(\ell+1)}{r^2\,C_{22}}\,;$$

$$q(r) \,=\, \alpha_{\gamma\mathrm{p}_0} \,-\, \frac{2}{r} \,+\, \frac{1}{r}\frac{C_{12}}{C_{22}} \,+\, \frac{\ell(\ell+1)}{r}\left(\frac{1}{r\,C_{22}}\right)' \,+\, \frac{\ell(\ell+1)}{r}\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\,;$$

$$\tilde{q}(r) \,=\, C_{11} \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{C_{12}}{r\,C_{22}} \,+\, \frac{\ell(\ell+1)}{r}\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\right]',$$

thus

$$r^2\,C_{22}\,\hat{q}(r) \,=\, -r^2\,C_{22} \,+\, \ell(\ell+1) \,=\, \frac{\sigma^2}{\mathrm{c}_0^2}r^2\,; \tag{A.2a}$$

$$r^2\,C_{22}\,q(r) \,=\, C_{22}(\alpha_{\gamma\mathrm{p}_0}r^2 \,-\, 2\,r) \,+\, r\,C_{12} \,+\, \ell(\ell+1)\Big(rC_{22}\left(\frac{1}{r\,C_{22}}\right)' \,+\, \frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big)\,; \tag{A.2b}$$

$$= C_{22}\big(\alpha_{\gamma\mathrm{p}_0}\,r^2 \,-\, 2\,r\big) \,+\, r\,C_{12} \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big)\,; \tag{A.2c}$$

$$r^2\,C_{22}\,\tilde{q}(r) \,=\, r^2C_{22}C_{11} \,+\, \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{22}\left(\frac{1}{rC_{22}}\right)'\right] \tag{A.2d}$$

$$+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{12} \tag{A.2e}$$

$$= r^2C_{22}C_{11} \,+\, \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) \,-\, \frac{2}{r^2} \,-\, \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\right] \tag{A.2f}$$

$$+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{12}\,. \tag{A.2g}$$

Note that in the formulation of $r^2\,C_{22}\,\tilde{q}(r)$ and $r^2\,C_{22}\,q(r)$, we have used

$$\left(\frac{1}{rC_{22}}\right)' \,=\, -\frac{C_{22} \,+\, r\,C_{22}'}{(rC_{22})^2} \quad\Rightarrow\quad r\,C_{22}\left(\frac{1}{r\,C_{22}}\right)' \,=\, -\frac{C_{22} \,+\, r\,C_{22}'}{r\,C_{22}} \,=\, -\frac{1}{r} \,-\, \frac{C_{22}'}{C_{22}}\,.$$

Let us first recall the proposition for clarity.

Proposition 4: For $r \leq r_a$, we have

1. For $\ell > 0$,

$$\frac{C_{22}'}{C_{22}} \,=\, -\frac{2}{r} \,+\, r^2\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, r\,\frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\,. \tag{A.3}$$

2. The coefficient of the first order term of the ODE (7.12) has the following form,

$$r\, C_{22}\, q(r) \;=\; -\alpha_{\gamma\mathrm{p}_0} \frac{\sigma^2}{\mathrm{c}_0^2}\, r \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2} \,-\, \ell(\ell+1)\, \frac{r\left(2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\right) \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}}{\frac{\sigma^2}{\mathrm{c}_0^2}\, r^2 \,-\, \ell(\ell+1)}\,, \tag{A.4}$$

or equivalently

$$\frac{r\, C_{22}\, q(r)}{k_0^2} \;=\; -\alpha_{\gamma\mathrm{p}_0}\, r \,+\, 2 \,-\, \ell(\ell+1)\, \frac{2\,\alpha_{\mathrm{c}_0}\, r \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\,\frac{1}{k_0^2}\, r \,+\, 2}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,. \tag{A.5}$$

3. The coefficient of the 0th- term of the ODE (7.12) has the form,

$$\begin{aligned} r^2\, C_{22}\, \tilde{q}(r) \;=\; & -\frac{\sigma^2}{\mathrm{c}_0^2}\,\frac{\left(-\sigma^2 \,+\, \Phi_0''\right)}{\mathrm{c}_0^2}\, r^2 \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0} r}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0}\, r \,-\, 1\right) \\ & -\, \ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \ell(\ell+1)\frac{\Phi_0'}{\mathrm{c}_0^2}\left(-\frac{\Phi_0'}{\mathrm{c}_0^2} \,+\, \alpha_{\rho_0}\right) \\ & -\, \ell(\ell+1)\left(2 \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\, r\right) \frac{r\left(2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\right) \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}}{\frac{\sigma^2}{\mathrm{c}_0^2}\, r^2 \,-\, \ell(\ell+1)}\,. \end{aligned} \tag{A.6}$$

A form entirely in terms of ρ_0, c_0 and Φ_0 , and k_0 is given as

$$\begin{aligned} \frac{r^2\, C_{22}\, \tilde{q}(r)}{k_0^2} \;=\; & \left(k_0^2 \,-\, \frac{\Phi_0''}{\mathrm{c}_0^2}\right) r^2 \,+\, 2\, r\left(\frac{\Phi_0'}{\mathrm{c}_0^2} \,-\, \alpha_{\rho_0} - 2\alpha_{\mathrm{c}_0}\right) \,-\, 2 - \ell(\ell+1) \\ & +\, \frac{\ell(\ell+1)}{k_0^2}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\left(\alpha_{\rho_0} \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2}\right) \\ & -\, \frac{\ell(\ell+1)}{k_0^2}\left(2 \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2}\, r\right) \frac{2\,\alpha_{\mathrm{c}_0}\, r \,+\, 2 \,+\, \mathrm{i}\,\frac{\omega}{k_0^2}\,\frac{(2\Gamma)'}{\mathrm{c}_0^2} r}{r^2 \,-\, \frac{\ell(\ell+1)}{k_0^2}}\,. \end{aligned} \tag{A.7}$$

Proof. **Statement 1** From the definition of C_{22} in (4.35), we have

$$C_{22}' \;=\; -\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)' \,-\, 2\frac{\ell(\ell+1)}{r^3}\,, \tag{A.8}$$

with

$$\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)' \;=\; 2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} \,+\, \frac{(\sigma^2)'}{\mathrm{c}_0^2}\,, \qquad (\sigma^2)' \;=\; \mathrm{i}\,\omega\,(2\Gamma)'\,. \tag{A.9}$$

Thus, we have

$$\frac{C_{22}'}{C_{22}} \;=\; \frac{1}{r}\,\frac{-2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0}\, r^3 \,-\, \frac{\mathrm{i}\,\omega\,(2\Gamma)'}{\mathrm{c}_0^2}\, r^3 \,-\, 2\ell(\ell+1)}{-\frac{\sigma^2}{\mathrm{c}_0^2} r^2 \,+\, \ell(\ell+1)}\,.$$

Next, we consider

$$\begin{aligned} \frac{C_{22}'}{C_{22}} \,+\, \frac{2}{r} \;&=\; \frac{1}{r}\,\frac{-2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0}\, r^3 \,-\, \frac{\mathrm{i}\,\omega\,(2\Gamma)'}{\mathrm{c}_0^2} r^3 \,-\, 2\frac{\sigma^2}{\mathrm{c}_0^2} r^2}{-\frac{\sigma^2}{\mathrm{c}_0^2} r^2 \,+\, \ell(\ell+1)} \\ &=\; r^2\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\, r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, r\,\frac{2\sigma^2}{\sigma^2\, r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\,, \end{aligned} \tag{A.10}$$

thus

$$-\frac{C_{22}'}{C_{22}} + \frac{1}{r} = \frac{3}{r} - r^2 \frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} - r\,\frac{2\sigma^2}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2}\,, \tag{A.11}$$

and

$$-\frac{C_{22}'}{C_{22}} - \frac{1}{r} = \frac{1}{r} - r^2 \frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} - r\,\frac{2\sigma^2}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2}\,. \tag{A.12}$$

Statement 2a We now consider $r^2\,C_{22}\,q(r)$. From its definition in (D.16),

$$\begin{aligned} r^2\,C_{22}\,q(r) &= C_{22}(\alpha_{\gamma\mathrm{p}_0} r^2 - 2\,r) + r\,C_{12} + \ell(\ell+1)\Big(r C_{22}\left(\frac{1}{r\,C_{22}}\right)' + \frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big) \\ &= \alpha_{\gamma\mathrm{p}_0}\,(C_{22}\,r^2) - 2\,r\,C_{22} + r\,C_{12} + \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big)\,. \end{aligned} \tag{A.13}$$

We substitute the definition of C_{12} on the right-hand side of (A.13),

$$\begin{aligned} C_{22} &= -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\ell(\ell+1)}{r^2}\,, \\ C_{12} &= \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{r\,\gamma} - \frac{\alpha_{\gamma\mathrm{p}_0}}{r} - \frac{1}{r^2}\right)\,, \end{aligned}$$

and using (A.11), we obtain

$$\begin{aligned} r^2\,C_{22}\,q(r) &= \alpha_{\gamma\mathrm{p}_0}\Big(-\frac{\sigma^2}{\mathrm{c}_0^2} r^2 + \ell(\ell+1)\Big) + \Big(2\frac{\sigma^2}{\mathrm{c}_0^2} r - 2\frac{\ell(\ell+1)}{r}\Big) \\ &\quad + \ell(\ell+1)\frac{\alpha_{\mathrm{p}_0} - \gamma\,\alpha_{\gamma\mathrm{p}_0}}{\gamma} - \frac{\ell(\ell+1)}{r} \\ &\quad + \frac{3\,\ell(\ell+1)}{r} - r^2\,\ell(\ell+1)\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} \\ &\quad - r\,\ell(\ell+1)\,\frac{2\sigma^2}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} - \ell(\ell+1)\frac{\alpha_{\mathrm{p}_0}}{\gamma} \\ &= -\alpha_{\gamma\mathrm{p}_0}\frac{\sigma^2}{\mathrm{c}_0^2} r^2 + 2\frac{\sigma^2}{\mathrm{c}_0^2} r + - r^2\,\ell(\ell+1)\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} \\ &\quad - r\,\ell(\ell+1)\,\frac{2\sigma^2}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2}\,. \end{aligned}$$

Statement 2b Let us now consider $r^2\,C_{22}\,\tilde{q}(r)$. From (D.16), we have

$$\begin{aligned} r^2\,C_{22}\,\tilde{q}(r) &= r^2 C_{22} C_{11} + \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right) r C_{12} \\ &\quad + \ell(\ell+1)\left[\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' + \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right) r C_{22}\left(\frac{1}{r C_{22}}\right)'\right] \end{aligned} \tag{A.14}$$

$$\begin{aligned} \Rightarrow r^2\,C_{22}\,\tilde{q}(r) &= r^2 C_{22} C_{11} + \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right) r C_{12} \\ &\quad + \ell(\ell+1)\left[\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) - \frac{2}{r^2} - \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\right]\,. \end{aligned} \tag{A.15}$$

We consider the first two terms of the right-hand side of (A.15),

$$I_1 := r^2\,C_{22}\,C_{11} + \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right) r C_{12}\,.$$

Recall from their definitions,

$$C_{12} \,=\, \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{r\,\gamma} \,-\, \frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,-\, \frac{1}{r^2}\right) \tag{A.16a}$$

$$C_{11} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,+\, 2\frac{\alpha_{\gamma\mathrm{p}_0}}{r} \,+\, \frac{2}{r^2} \,-\, 2\frac{\alpha_{\mathrm{p}_0}}{r\gamma} \quad\Rightarrow C_{11} \,=\, -\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,-\, \frac{2}{\ell(\ell+1)}C_{12}\,. \tag{A.16b}$$

This gives

$$\begin{aligned} r^2\,C_{22}\,C_{11} \,&=\, \left(-\frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \ell(\ell+1)\right)\,\left(-\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2} \,-\, \frac{2}{\ell(\ell+1)}C_{12}\right) \\ &=\, \left(-\frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \ell(\ell+1)\right)\,\left(-\frac{\sigma^2}{\mathrm{c}_0^2} \,+\, \frac{\Phi_0''}{\mathrm{c}_0^2}\right) \,+\, \frac{\sigma^2}{\mathrm{c}_0^2}r^2\frac{2}{\ell(\ell+1)}C_{12} \,-\, 2\,C_{12}\,. \end{aligned}$$

$$\begin{aligned} \Rightarrow I_1 \,&=\, r^2\,C_{22}\,C_{11} \,+\, 2C_{12} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\,C_{12} \\ &=\left(-\frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \ell(\ell+1)\right)\,\frac{-\sigma^2 \,+\, \Phi_0''}{\mathrm{c}_0^2} \,+\, \frac{\sigma^2}{\mathrm{c}_0^2}r^2\frac{2}{\ell(\ell+1)}C_{12} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\,C_{12}\,. \end{aligned}$$

$$\begin{aligned} \Rightarrow I_1 = &\left(-\frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \ell(\ell+1)\right)\,\frac{-\sigma^2 \,+\, \Phi_0''}{\mathrm{c}_0^2} \,+\, 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0}r}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0}\,r \,-\, 1\right) \\ &\qquad -\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\,\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0} \,-\, \frac{1}{r}\right)\,. \end{aligned} \tag{A.17}$$

We consider the remaining term on the right-hand side of (A.15),

$$I_2 \,:=\, \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) \,-\, \frac{2}{r^2} \,-\, \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\right]\,.$$

Using (A.12), we obtain

$$\begin{aligned} &\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) \,-\, \frac{2}{r^2} \\ &\qquad = \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(\frac{1}{r} \,-\, r^2\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,-\, r\,\frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\right) \,-\, \frac{2}{r^2} \\ &\qquad = -\left(r\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, \frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\right)\left(2 \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right) \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\frac{1}{r}\,, \end{aligned}$$

and we get

$$\begin{aligned} I_2 \,=\, &-\,\ell(\ell+1)\left(r\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, \frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\right)\left(2 \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right) \\ &-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\frac{\ell(\ell+1)}{r} \,-\, \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\,. \end{aligned} \tag{A.18}$$

Finally, putting together (A.17) and (A.18), we obtain the expression,

$$\begin{aligned} &r^2\,C_{22}\,\tilde{q}(r) \\ &\;=\left(-\frac{\sigma^2}{\mathrm{c}_0^2}r^2 \,+\, \ell(\ell+1)\right)\,\frac{-\sigma^2 \,+\, \Phi_0''}{\mathrm{c}_0^2} \,-\, \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' \\ &\quad +\, 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0}r}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0}\,r \,-\, 1\right) \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma} \,-\, \alpha_{\gamma\mathrm{p}_0}\right) \\ &\quad -\,\ell(\ell+1)\left(r\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} \,+\, \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2} \,+\, \frac{2\sigma^2}{\sigma^2\,r^2 \,-\, \ell(\ell+1)\mathrm{c}_0^2}\right)\left(2 \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right)\,. \end{aligned} \tag{A.19}$$

Using (6.30), we have

$$\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' = \frac{\Phi_0''}{\mathrm{c}_0^2} + 2\frac{\Phi_0'}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0}\,, \tag{A.20}$$

and the above expression simplifies to

$$\begin{aligned} &r^2\, C_{22}\, \tilde{q}(r) \\ &= -\frac{\sigma^2}{\mathrm{c}_0^2}\frac{(-\sigma^2 + \Phi_0'')}{\mathrm{c}_0^2}\,r^2 + \ell(\ell+1)\frac{-\sigma^2 - 2\,\Phi_0'\,\alpha_{\mathrm{c}_0}}{\mathrm{c}_0^2} + 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0} r}{\gamma} - \alpha_{\gamma\mathrm{p}_0}\,r - 1\right) \\ &- \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma} - \alpha_{\gamma\mathrm{p}_0}\right) \\ &- \ell(\ell+1)\left(r\,\frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2} + \frac{2\sigma^2}{\sigma^2\,r^2 - \ell(\ell+1)\mathrm{c}_0^2}\right)\left(2 - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right). \end{aligned} \tag{A.21}$$

After rearrangement, we arrive at

$$r^2\, C_{22}\, \tilde{q}(r) \tag{A.22a}$$

$$= -\frac{\sigma^2}{\mathrm{c}_0^2}\frac{(-\sigma^2 + \Phi_0'')}{\mathrm{c}_0^2}\,r^2 + 2\frac{\sigma^2}{\mathrm{c}_0^2}\left(\frac{\alpha_{\mathrm{p}_0} r}{\gamma} - \alpha_{\gamma\mathrm{p}_0}\,r - 1\right) \tag{A.22b}$$

$$+ \ell(\ell+1)\frac{-\sigma^2 - 2\,\Phi_0'\,\alpha_{\mathrm{c}_0}}{\mathrm{c}_0^2} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma} - \alpha_{\gamma\mathrm{p}_0}\right) \tag{A.22c}$$

$$-\ell(\ell+1)\left(2 - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,r\right)\frac{r\left(2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,\frac{(2\Gamma)'}{\mathrm{c}_0^2}\right) + 2\frac{\sigma^2}{\mathrm{c}_0^2}}{\frac{\sigma^2}{\mathrm{c}_0^2}\,r^2 - \ell(\ell+1)}. \tag{A.22d}$$

We can further regroup the second expression (A.22c). Using (6.28a)

$$\alpha_{\mathrm{p}_0} = +\,\Phi_0'\,\frac{\gamma}{\mathrm{c}_0^2}\,, \tag{A.23}$$

and

$$\alpha_{\gamma\mathrm{p}_0} = 2\alpha_{\mathrm{c}_0} + \alpha_{\rho_0} \quad , \quad \text{cf. (6.28b)}, \tag{A.24}$$

we rewrite

$$\begin{aligned} \text{(A.22c)} &= \ell(\ell+1)\frac{-\sigma^2 - 2\,\Phi_0'\,\alpha_{\mathrm{c}_0}}{\mathrm{c}_0^2} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{\gamma} - \alpha_{\gamma\mathrm{p}_0}\right) \\ &= -\,\ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} + \ell(\ell+1)\left(\frac{-\,2\,\Phi_0'\,\alpha_{\mathrm{c}_0}}{\mathrm{c}_0^2} - \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)^2 + \frac{\alpha_{\mathrm{p}_0}\,\alpha_{\gamma\mathrm{p}_0}}{\gamma}\right) \\ &= -\,\ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} + \ell(\ell+1)\left(\frac{-\,2\,\Phi_0'\,\alpha_{\mathrm{c}_0}}{\mathrm{c}_0^2} - \left(\frac{\Phi_0'}{\mathrm{c}_0^2}\right)^2 + \alpha_{\gamma\mathrm{p}_0}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\right) \\ &= -\,\ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} + \ell(\ell+1)\left(-\left(\frac{\Phi_0'}{\mathrm{c}_0^2}\right)^2 + \alpha_{\rho_0}\,\frac{\Phi_0'}{\mathrm{c}_0^2}\right) \\ &= -\,\ell(\ell+1)\frac{\sigma^2}{\mathrm{c}_0^2} + \ell(\ell+1)\frac{\Phi_0'}{\mathrm{c}_0^2}\left(-\frac{\Phi_0'}{\mathrm{c}_0^2} + \alpha_{\rho_0}\right). \end{aligned} \tag{A.25}$$

□

Remark 22. *In the case where $\ell = 0$, we have*

$$\frac{C_{22}'}{C_{22}} = \frac{-2\,\frac{\sigma^2}{\mathrm{c}_0^2}\,\alpha_{\mathrm{c}_0} - \frac{\mathrm{i}\,\omega\,(2\Gamma)'}{\mathrm{c}_0^2}}{-\frac{\sigma^2}{\mathrm{c}_0^2}} = \frac{2\,\sigma^2\,\alpha_{\mathrm{c}_0} + \mathrm{i}\,\omega\,(2\Gamma)'}{\sigma^2}\,.$$

Therefore, this is a regular function. However, we do not need this since this term does not appear at all in the equation for $\ell = 0$. △

B Explicit expressions for the coefficients of the radial ODE in the atmosphere

In this appendix, we prove Proposition 5. We recall

$$C_{11} = -\frac{\sigma^2}{c_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} + \frac{\Phi_0''}{c_0^2}; \tag{B.1a}$$

$$C_{12} = \ell(\ell+1)\left(-\frac{\alpha_{\rho_0}}{r} + \frac{\alpha_{\rho_0}}{r\,\gamma} - \frac{1}{r^2}\right); \tag{B.1b}$$

$$C_{22} = -\frac{\sigma^2}{c_0^2} - \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\Phi_0'}{c_0^2}\frac{1}{r}. \tag{B.1c}$$

$$r^2\,C_{22}\,\hat{q}(r) = -r^2\,C_{22} + \ell(\ell+1) = \frac{\sigma^2}{c_0^2}r^2 + \frac{\alpha_{\rho_0}}{\gamma}r - \frac{\Phi_0'}{c_0^2}r; \tag{B.2a}$$

$$r^2\,C_{22}\,q(r) = C_{22}(\alpha_{\rho_0}\,r^2 - 2\,r) + r\,C_{12} + \ell(\ell+1)\Big(rC_{22}\left(\frac{1}{r\,C_{22}}\right)' + \frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\Big); \tag{B.2b}$$

$$= C_{22}(\alpha_{\rho_0}\,r^2 - 2\,r) + r\,C_{12} + \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} - \frac{\alpha_{\rho_0}}{\gamma}\Big); \tag{B.2c}$$

$$r^2\,C_{22}\,\tilde{q}(r) = r^2C_{22}C_{11} + \ell(\ell+1)\Big[\big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)' \tag{B.2d}$$

$$+ \big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)rC_{22}\big(\frac{1}{rC_{22}}\big)'\Big] + \big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)rC_{12} \tag{B.2e}$$

$$= r^2C_{22}C_{11} + \ell(\ell+1)\left[\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) - \frac{2}{r^2}\right] \tag{B.2f}$$

$$+ \left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)rC_{12}. \tag{B.2g}$$

We have introduced (4.57),

$$E_{\mathrm{he}} := -\frac{\alpha_{\rho_0}}{\gamma} + \frac{\Phi_0'}{c_0^2}, \tag{B.3}$$

and in (4.50),

$$k_0 = \frac{\sigma}{c_0}, \tag{B.4}$$

Proposition 4 In $r \geq r_a$

$$r^2\,C_{22}\,\hat{q}(r) = r(k_0^2\,r - E_{\mathrm{he}}). \tag{B.5}$$

$$r^2\,C_{22}\,q(r) = (\alpha_{\rho_0}r - 2)\,(-k_0^2\,r + E_{\mathrm{he}}) + \ell(\ell+1)\frac{(k_0^2)'\,r^2 + 2\,k_0^2\,r - (E_{\mathrm{he}} + r\frac{\Phi_0''}{c_0^2})}{-k_0^2r^2 + \ell(\ell+1) + rE_{\mathrm{he}}}. \tag{B.6}$$

$$\begin{aligned} r^2\,C_{22}\,\tilde{q}(r) &\\ = (k_0^2\,r^2 - rE_{\mathrm{he}})&\left(k_0^2 - \frac{\Phi_0''}{c_0^2}\right) + 2\,(k_0^2r^2 - rE_{\mathrm{he}})\left(-\frac{\alpha_{\rho_0}}{r} + \frac{\alpha_{\rho_0}}{r\,\gamma} - \frac{1}{r^2}\right)\\ + \ell(\ell+1)&\left(-k_0^2 + \frac{\Phi_0''}{c_0^2}\right) - \ell(\ell+1)\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} - \alpha_{\rho_0}\right)\\ + \ell(\ell+1)&\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 + 2\,k_0^2\,r - \left(E_{\mathrm{he}} + r\frac{\Phi_0''}{c_0^2}\right)}{-k_0^2r^2 + \ell(\ell+1) + rE_{\mathrm{he}}}. \end{aligned} \tag{B.7}$$

Proof. In the new notation,

$$r^2\, C_{22}\, \hat{q}(r) \;=\; -r^2\, C_{22} \,+\, \ell(\ell+1) \;=\; k_0^2\, r^2 \,+\, \frac{\alpha_{\rho_0}}{\gamma} r \,-\, \frac{\Phi_0'}{\mathrm{c}_0^2} r \tag{B.8}$$

can be written as

$$r^2\, C_{22}\, \hat{q}(r) \;\;=\; k_0^2\, r^2 \;-\; r\, E_{\mathrm{he}}\,. \tag{B.9}$$

Component C_{22} and its derivative Recall from (D.20), the definition of C_{22},

$$C_{22} \;=\; -\frac{\sigma^2}{\mathrm{c}_0^2} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} \,+\, \frac{\ell(\ell+1)}{r^2} + \frac{\Phi_0'}{\mathrm{c}_0^2}\frac{1}{r}\,. \tag{B.10}$$

In terms of E_{he}, C_{22} is written as,

$$C_{22} \;=\; -k_0^2 \,+\, \frac{\ell(\ell+1)}{r^2} + \frac{E_{\mathrm{he}}(r)}{r} \tag{B.11}$$

We can write

$$C_{22}' \;=\; -(k_0^2)' \,-\, 2\frac{\ell(\ell+1)}{r^3} \,-\, \frac{E_{\mathrm{he}}}{r^2} \,+\, \frac{E_{\mathrm{he}}'}{r}\,. \tag{B.12}$$

We note that

$$E_{\mathrm{he}}' \;=\; \frac{\Phi_0''}{\mathrm{c}_0^2}\,. \tag{B.13}$$

We consider

$$\frac{C_{22}'}{C_{22}} = \frac{-(k_0^2)'}{C_{22}} - \frac{2\ell(\ell+1)\, r^{-1}}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \,+\, \frac{-E_{\mathrm{he}} + r E_{\mathrm{he}}'}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}}\,. \tag{B.14}$$

Thus

$$\begin{aligned}
-\frac{C_{22}'}{C_{22}} \,-\, \frac{2}{r} &= -\frac{-(k_0^2)'}{C_{22}} + \frac{2}{r}\,\frac{\ell(\ell+1) - (-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \,+\, \frac{E_{\mathrm{he}} - r E_{\mathrm{he}}'}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \\
&= -\frac{-(k_0^2)'}{C_{22}} + \frac{2}{r}\,\frac{k_0^2 r^2 - r\, E_{\mathrm{he}}}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \,+\, \frac{E_{\mathrm{he}} - r E_{\mathrm{he}}'}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \\
&= -\frac{-(k_0^2)'}{C_{22}} \,+\, \frac{2\, k_0^2 r}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}} \,+\, \frac{-E_{\mathrm{he}} - r E_{\mathrm{he}}'}{-k_0^2 r^2 \,+\, \ell(\ell+1) + r\, E_{\mathrm{he}}}\,.
\end{aligned} \tag{B.15}$$

We can thus write

$$-\frac{C_{22}'}{C_{22}} \,-\, \frac{2}{r} = \frac{(k_0^2)'\, r^2 \,+\, 2\, k_0^2\, r \,-\, (E_{\mathrm{he}} \,+\, r E_{\mathrm{he}}')}{-k_0^2 r^2 \,+\, \ell(\ell+1) \,+\, r E_{\mathrm{he}}}\,. \tag{B.16}$$

Statement 1 We start with expression of $r^2\, C_{22}\, q$ given in (B.2),

$$r^2\, C_{22}\, q \;=\; C_{22}(\alpha_{\rho_0}\, r^2 \,-\, 2\, r) \,+\, r\, C_{12} \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\Big)\,. \tag{B.17}$$

Using its definition in (D.20)

$$C_{12} \;=\; \ell(\ell+1)\, \left(-\frac{\alpha_{\rho_0}}{r} \,+\, \frac{\alpha_{\rho_0}}{r\,\gamma} \,-\, \frac{1}{r^2}\right) \tag{B.18}$$

we have

$$\begin{aligned}
&r C_{12} \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\Big) \\
&= \ell(\ell+1)\, \left(-\alpha_{\rho_0} \,+\, \frac{\alpha_{\rho_0}}{\gamma} \,-\, \frac{1}{r}\right) \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\Big) \\
&= -\ell(\ell+1)\left(\alpha_{\rho_0} \,+\, \frac{C_{22}'}{C_{22}}\right)\,.
\end{aligned} \tag{B.19}$$

With the above calculation, expression (B.17) simplifies to,

$$r^2\, C_{22}\, q(r) \;=\; C_{22}(\alpha_{\rho_0}\, r^2 \,-\, 2\,r) \;-\; \ell(\ell+1)\alpha_{\rho_0} \;-\; \ell(\ell+1)\frac{C_{22}'}{C_{22}}\,. \tag{B.20}$$

We can rewrite the first two terms as

$$\begin{aligned}
&C_{22}(\alpha_{\rho_0}\, r^2 \,-\, 2\,r) \;-\; \ell(\ell+1)\alpha_{\rho_0}\\
&= \big(-\alpha_{\rho_0}\, k_0^2\, r^2 \;+\; \alpha_{\rho_0}\ell(\ell+1) + \alpha_{\rho_0} r E_{\mathrm{he}}\big) \;-2\,r\left(-k_0^2 \;+\; \frac{\ell(\ell+1)}{r^2} + \frac{E_{\mathrm{he}}}{r}\right) \;-\; \ell(\ell+1)\alpha_{\rho_0}\\
&= -\alpha_{\rho_0}\, k_0^2\, r^2 \;+\; \alpha_{\rho_0}\, r\, E_{\mathrm{he}} \;+\; 2\,r\,k_0^2 \;-\; 2\frac{\ell(\ell+1)}{r} \;-\; 2E_{\mathrm{he}}\\
&= \alpha_{\rho_0} r(-k_0^2\, r \;+\; E_{\mathrm{he}}) + 2(k_0^2\, r - E_{\mathrm{he}}) \;-\; 2\frac{\ell(\ell+1)}{r}\\
&= (\alpha_{\rho_0} r \;-\; 2)\,(-k_0^2\, r \;+\; E_{\mathrm{he}}) \;-\; 2\frac{\ell(\ell+1)}{r}\,.
\end{aligned} \tag{B.21}$$

Together with identity (B.16), we obtain the following expression,

$$\begin{aligned}
r^2\, C_{22}\, q(r) &= (\alpha_{\rho_0} r \,-\, 2)\,(-k_0^2\, r \,+\, E_{\mathrm{he}}) \;-\; \ell(\ell+1)\left(\frac{2}{r} + \frac{C_{22}'}{C_{22}}\right)\\
&= (\alpha_{\rho_0} r \,-\, 2)\,(-k_0^2\, r \,+\, E_{\mathrm{he}}) \;+\; \ell(\ell+1)\frac{(k_0^2)'\, r^2 \;+\; 2\,k_0^2\, r \;-\; (E_{\mathrm{he}} \,+\, rE_{\mathrm{he}}')}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}\,.
\end{aligned} \tag{B.22}$$

Statement 2 From (D.23)

$$r^2\, C_{22}\, \tilde{q}(r) \;=\; r^2 C_{22} C_{11} \;+\; \ell(\ell+1)\left[\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) - \frac{2}{r^2}\right] + \left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right) rC_{12}\,. \tag{B.23}$$

Break the right-hand-side into

$$I_1 \;:=\; r^2\, C_{22}\, C_{11} \;+\; 2C_{12} \;-\; \frac{\alpha_{\mathrm{p}_0}}{\gamma}\, r\, C_{12}\,; \tag{B.24a}$$

$$I_2 \;:=\; \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\rho_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) \,-\, \frac{2}{r^2}\right]. \tag{B.24b}$$

We have

$$\begin{aligned}
r^2\, C_{22}\, C_{11} \;&=\; (-k_0^2\, r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}})\,\left(-k_0^2 \;+\; \frac{\Phi_0''}{c_0^2} \;-\; \frac{2}{\ell(\ell+1)}C_{12}\right)\\
&=\; (-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}})\,\left(-k_0^2 \;+\; \frac{\Phi_0''}{c_0^2}\right) \;+\; (k_0^2 r^2 - rE_{\mathrm{he}})\,\frac{2}{\ell(\ell+1)}C_{12} \;-\; 2\,C_{12}\,.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\; I_1 \;&:=\; r^2\, C_{22}\, C_{11} \;+\; 2C_{12} \;-\; \frac{\alpha_{\mathrm{p}_0}}{\gamma}\, r\, C_{12}\\
&= (-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}})\,\left(-k_0^2 \;+\; \frac{\Phi_0''}{c_0^2}\right) \;+\; (k_0^2 r^2 - rE_{\mathrm{he}})\,\frac{2}{\ell(\ell+1)}C_{12} \;-\; \frac{\alpha_{\mathrm{p}_0}}{\gamma}\, r\, C_{12}\,.
\end{aligned}$$

Substitute in the definition of C_{12}, we obtain

$$\begin{aligned}
I_1 &= (-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}})\,\left(-k_0^2 \;+\; \frac{\Phi_0''}{c_0^2}\right)\\
&+\, 2\,(k_0^2 r^2 - rE_{\mathrm{he}})\,\left(-\frac{\alpha_{\rho_0}}{r} \;+\; \frac{\alpha_{\rho_0}}{r\,\gamma} \;-\; \frac{1}{r^2}\right) \;-\; \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\,\left(-\alpha_{\rho_0} \;+\; \frac{\alpha_{\rho_0}}{\gamma} \;-\; \frac{1}{r}\right).
\end{aligned} \tag{B.25}$$

We next consider I_2. Using (B.16), we can write

$$-\frac{C_{22}'}{C_{22}} \;-\; \frac{1}{r} \;=\; \frac{1}{r} \;+\; \frac{(k_0^2)'\, r^2 \;+\; 2\,k_0^2\, r \;-\; (E_{\mathrm{he}} \,+\, rE_{\mathrm{he}}')}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}\,. \tag{B.26}$$

Thus

$$\frac{1}{\ell(\ell+1)} I_2 = \left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\left(\frac{1}{r} + \frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; (E_{\mathrm{he}} + rE'_{\mathrm{he}})}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}\right) - \frac{2}{r^2}$$
$$= \left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; (E_{\mathrm{he}} + rE'_{\mathrm{he}})}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}} - \frac{1}{r}\frac{\alpha_{\rho_0}}{\gamma}. \tag{B.27}$$

Putting together expression (B.25) for I_1 and (B.27) for I_2,

$$r^2\, C_{22}\, \tilde{q}(r)$$
$$= \left(-k_0^2 r^2 \,+\, \ell(\ell+1) \,+\, rE_{\mathrm{he}}\right)\left(-k_0^2 \,+\, \frac{\Phi_0''}{c_0^2}\right)$$
$$+\, 2\left(k_0^2 r^2 - rE_{\mathrm{he}}\right)\left(-\frac{\alpha_{\rho_0}}{r} + \frac{\alpha_{\rho_0}}{r\,\gamma} - \frac{1}{r^2}\right) - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\,\ell(\ell+1)\left(-\alpha_{\rho_0} + \frac{\alpha_{\rho_0}}{\gamma}\right) \tag{B.28}$$
$$+\,\ell(\ell+1)\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; (E_{\mathrm{he}} + rE'_{\mathrm{he}})}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}.$$

After rearrangement we obtain

$$r^2\, C_{22}\, \tilde{q}(r)$$
$$= \left(k_0^2\, r^2 \,-\, rE_{\mathrm{he}}\right)\left(k_0^2 \,-\, \frac{\Phi_0''}{c_0^2}\right) \,+\, 2\left(k_0^2 r^2 - rE_{\mathrm{he}}\right)\left(-\frac{\alpha_{\rho_0}}{r} + \frac{\alpha_{\rho_0}}{r\,\gamma} - \frac{1}{r^2}\right)$$
$$+\,\ell(\ell+1)\left(-k_0^2 + \frac{\Phi_0''}{c_0^2}\right) - \ell(\ell+1)\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} - \alpha_{\rho_0}\right) \tag{B.29}$$
$$+\,\ell(\ell+1)\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; \left(E_{\mathrm{he}} \,+\, r\frac{\Phi_0''}{c_0^2}\right)}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}.$$

□

We next consider $\mathfrak{h}$ and $\mathfrak{g}$.

Proposition 8 : For $r \geq r_a$, we have

$$\mathfrak{h} = \alpha_{\rho_0} \,-\, \frac{2}{r} \,+\, \ell(\ell+1)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; E_{\mathrm{he}} \;-\; r\frac{\Phi_0''}{c_0^2}}{\left(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\mathrm{he}}\right)\,\left(k_0^2\, r^2 \;-\; r\,E_{\mathrm{he}}\right)}, \tag{B.30}$$

$$\mathfrak{g} = -k_0^2 \,+\, \frac{2\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right)}{r} \,+\, \frac{2}{r^2} \,+\, \ell(\ell+1)\frac{k_0^2 \,+\, \frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} - \alpha_{\rho_0}\right) \,-\, \frac{\Phi_0''}{c_0^2}}{k_0^2\, r^2 \;-\; r\,E_{\mathrm{he}}} \tag{B.31a}$$

$$+\,\frac{\Phi_0''}{c_0^2} \,+\, \ell(\ell+1)\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; E_{\mathrm{he}} \;-\; r\frac{\Phi_0''}{c_0^2}}{\left(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\mathrm{he}}\right)\,\left(k_0^2\, r^2 \;-\, r\,E_{\mathrm{he}}\right)}. \tag{B.31b}$$

Under the hypothesis of constant attenuation,

$$\mathfrak{h}' = \frac{2}{r^2} \,+\, \ell(\ell+1)\frac{2\,k_0^2 \;-\; \frac{4\pi G}{c_0^2}r\,\rho_0' \;-\; 2\frac{\Phi_0'}{c_0^2}\frac{1}{r}}{\left(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\mathrm{he}}\right)\,\left(k_0^2\, r^2 \;-\; r\,E_{\mathrm{he}}\right)},$$
$$-\,\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\mathrm{he}} \;-\; r\frac{\Phi_0''}{c_0^2}\right)\left(2\,k_0^2\,r \;-\; E_{\mathrm{he}} \;-\; r\frac{\Phi_0''}{c_0^2}\right)\left(2k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; 2rE_{\mathrm{he}}\right)}{\left(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\mathrm{he}}\right)^2\,\left(2\,k_0^2\, r^2 \;-\; r\,E_{\mathrm{he}}\right)^2} \tag{B.32}$$

Proof.

$$-\frac{r^2\, C_{22}\, q(r)}{r^2\, C_{22}\, \hat{q}(r)} = -\frac{\left(\alpha_{\rho_0} r \,-\, 2\right)\left(-k_0^2\, r \,+\, E_{\mathrm{he}}\right)}{r\left(k_0^2\, r \,-\, E_{\mathrm{he}}\right)}$$
$$-\,\ell(\ell+1)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; \left(E_{\mathrm{he}} \,+\, r\frac{\Phi_0''}{c_0^2}\right)}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\mathrm{he}}}\,\frac{1}{r\left(k_0^2\, r \,-\, E_{\mathrm{he}}\right)}. \tag{B.33}$$

Thus

$$\mathfrak{h} \;=\; \alpha_{\rho_0} \;-\; \frac{2}{r} \;+\; \ell(\ell+1)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\,.. \tag{B.34}$$

We obtain readily the expression for $\mathfrak{g}$.

$$\begin{aligned}
\mathfrak{g} \;&=\; -\frac{r^2\,C_{22}\,\tilde{q}(r)}{r^2\,C_{22}\,\hat{q}(r)} \\
&=\; -k_0^2 \;+\; \frac{\Phi_0''}{\mathrm{c}_0^2} \;+\; 2\left(\frac{\alpha_{\rho_0}}{r} \;-\; \frac{\alpha_{\rho_0}}{r\,\gamma} \;+\; \frac{1}{r^2}\right) \\
&+\; \ell(\ell+1)\frac{k_0^2 - \frac{\Phi_0''}{\mathrm{c}_0^2}}{r(k_0^2\,r \;-\; E_{\rm he})} \;+\; \ell(\ell+1)\frac{\alpha_{\rho_0}}{\gamma}\left(\frac{\alpha_{\rho_0}}{\gamma} \;-\; \alpha_{\rho_0}\right)\frac{1}{r(k_0^2\,r \;-\; E_{\rm he})} \\
&-\ell(\ell+1)\left(\frac{2}{r} \;-\; \frac{\alpha_{\rho_0}}{\gamma}\right)\frac{(k_0^2)'\,r^2 \;+\; 2\,k_0^2\,r \;-\; \left(E_{\rm he} \;+\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{-k_0^2 r^2 \;+\; \ell(\ell+1) \;+\; rE_{\rm he}}\,\frac{1}{r(k_0^2\,r \;-\; E_{\rm he})}\,.
\end{aligned} \tag{B.35}$$

The final expression is obtained from rearrangement.

Derivative of $\mathfrak{h}$ To calculate $\mathfrak{h}'$, for simplicity, we assume constant attenuation so $(k_0^2)' = 0$.

$$\begin{aligned}
\mathfrak{h}' = &-\left(\frac{2}{r}\right)' \;+\; \ell(\ell+1)\frac{\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)'}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})} \\
&-\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)^2\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)' \\
&-\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})^2}(k_0^2\,r^2 \;-\; r\,E_{\rm he})'
\end{aligned} \tag{B.36}$$

We next recall that $E_{\rm he}' = \frac{\Phi_0''}{\mathrm{c}_0^2}$, and

$$\Phi_0''' \;=\; 4\pi G\rho_0' + \frac{2}{r^2}\Phi_0' - \frac{2}{r}\Phi_0''\,. \tag{B.37}$$

$$\begin{aligned}
\Rightarrow E_{\rm he}' \;+\; \left(r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)' \;&=\; \frac{\Phi_0''}{\mathrm{c}_0^2} + \frac{\Phi_0''}{\mathrm{c}_0^2} + \frac{r}{\mathrm{c}_0^2}\left(4\pi G\rho_0' + \frac{2}{r^2}\Phi_0' - \frac{2}{r}\Phi_0''\right) \\
&=\; \frac{4\pi G}{\mathrm{c}_0^2}r\,\rho_0' + \;2\frac{\Phi_0'}{\mathrm{c}_0^2}\frac{1}{r}\,.
\end{aligned} \tag{B.38}$$

$$\begin{aligned}
\mathfrak{h}' \;=\; &\frac{2}{r^2} \;+\; \ell(\ell+1)\frac{2\,k_0^2 \;-\; \frac{4\pi G}{\mathrm{c}_0^2}r\,\rho_0' \;-\; 2\frac{\Phi_0'}{\mathrm{c}_0^2}\frac{1}{r}}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\,, \\
&-\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)^2\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\big(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\big) \\
&-\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\right)}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})^2}\big(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{\mathrm{c}_0^2}\big)
\end{aligned} \tag{B.39}$$

The last two terms are grouped using,

$$\frac{1}{k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}} \;+\; \frac{1}{k_0^2\,r^2 \;-\; r\,E_{\rm he}} \;=\; \frac{2k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; 2rE_{\rm he}}{(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he})\,(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\,. \tag{B.40}$$

We thus arrive at the stated expression for $\mathfrak{h}'$,

$$\begin{aligned}\mathfrak{h}' \;=\; & \frac{2}{r^2} \;+\; \ell(\ell+1)\frac{2\,k_0^2 \;-\; \frac{4\pi G}{c_0^2} r\,\rho_0' \;-\; 2\frac{\Phi_0'}{c_0^2}\frac{1}{r}}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)\;(k_0^2\,r^2 \;-\; r\,E_{\rm he})}\,, \\ & -\;\frac{\ell(\ell+1)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{c_0^2}\right)\left(2\,k_0^2\,r \;-\; E_{\rm he} \;-\; r\frac{\Phi_0''}{c_0^2}\right)\left(2k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; 2rE_{\rm he}\right)}{\big(k_0^2 r^2 \;-\; \ell(\ell+1) \;-\; rE_{\rm he}\big)^2\;(\,k_0^2\,r^2 \;-\; r\,E_{\rm he})^2}\,.\end{aligned} \tag{B.41}$$

□

C Results of the solutions of ODE

Here we cite the results in [13] needed for the construction of solution in Section 9. There are two types of results, one for interval on which the coefficients of the ODE are continuous, and a second result for regular singular ODE.

First, we put together the results of the Theorems 1 and 3 of [13], which give the existence and uniqueness of the initial boundary value problem for an ODE with continuous coefficients.

Theorem 2 (Theorems 1 and 3 p.103 of [13]). *Let q_1 and q_0 be continuous functions on an interval I containing x_0. For any set of constants $(c_0, c_1) \in \mathbb{C}^2$, there exists a unique solution u of*

$$u'' \;+\; q_1(r)\,u' \;+\; q_0(r) \;=\; 0 \tag{C.1}$$

on the entire interval I satisfying

$$u(x_0) \;=\; c_0\,, \qquad u'(x_0) \;=\; c_1\,. \tag{C.2}$$

When there are singularities, we suppose that they are of regular singular type and cite the results of Section 6 of [13].

Next, we use the Theorems 3 and 4 of [13].

Theorem 3 (Theorem 3 p. 158 and Theorem 4 p. 165 of [13]). *Consider the equation*

$$r^2 u'' \;+\; p(r)\,r\,u' \;+\; q(r)u \;=\; 0\,, \qquad r > 0 \tag{C.3}$$

where p and q have power series expansions which are convergent for $|r| < r_0$ with $r_0 > 0$. Let λ_- and λ_+ with

$$\mathrm{Re}\,\lambda_- \;\leq\; \mathrm{Re}\,\lambda_+\,, \tag{C.4}$$

the two indicial roots of the indicial polynomial

$$\lambda\,(\lambda \;-\; 1) \;+\; p(0)\,\lambda \;+\; q(0) \;=\; 0\,. \tag{C.5}$$

1. *If $\lambda_+ - \lambda_- \notin \mathbb{Z}$, there are two linearly independent solutions u_1 and u_2 of the form*

$$u_1(r) \;=\; r^{\lambda_+}\sum_{k=0}^{\infty} c_k r^k\,, \qquad c_0 = 1\,, \tag{C.6a}$$

$$u_2(r) \;=\; r^{\lambda_-}\sum_{k=0}^{\infty} \tilde{c}_k r^k\,, \qquad \tilde{c}_0 = 1\,, \tag{C.6b}$$

where the series converges for $|r| \leq r_0$.

2. *If $\lambda_- = \lambda_+$, two linearly independent solutions are given by*

$$u_1(r) \;=\; r^{\lambda_+}\sum_{k=0}^{\infty} c_k r^k\,, \qquad c_0 \neq 0\,; \tag{C.7a}$$

$$u_2(r) \;=\; r^{\lambda_+ + 1}\sum_{k=0}^{\infty} \tilde{c}_k r^k \;+\; u_1\,\log r\,. \tag{C.7b}$$

3. If $\lambda_+ - \lambda_-$ is a positive integer, then two linearly independent solutions are given by

$$u_1(r) \;=\; r^{\lambda_+} \sum_{k=0}^{\infty} c_k r^k \,, \qquad c_0 \neq 0\,, \tag{C.8a}$$

$$u_2(r) \;=\; r^{\lambda_-} \sum_{k=0}^{\infty} \tilde{c}_k r^k \;+\; c\, u_1 \,\log r\,, \qquad \tilde{c}_0 \neq 0\,, \tag{C.8b}$$

with a constant c that can be zero.

D Alternative method for computation of the vectorial quantities

D.1 Discussion on the hierarchy of the background parameters

While (6.12) and (6.20) give a way to obtain higher derivatives of p_0 and Φ_0, it is not enough to calculate their scale height functions, if we do not assume a priori knowledge of α'_{ρ_0} and of α'_{c_0}. For the moment, we assume that we are only given

$$\mathrm{c}_0\,,\; \rho_0\,,\; \alpha_{\mathrm{c}_0}\,,\; \alpha_{\rho_0}(=\alpha)\,. \tag{D.1}$$

1. We can use (6.20) to calculate p'_0, with

$$\mathrm{p}'_0 \;=\; 4\pi G\,\frac{\rho_0(r)}{r^2}\int_0^r \rho_0(s)\,s^2\,ds\,. \tag{D.2}$$

   The second-order derivative is then obtained from (6.15b),

$$\mathrm{p}''_0 \;=\; -\left(\frac{2}{r} \,+\, \alpha_{\rho_0}\right)\,\mathrm{p}'_0 \;+\; 4\,\pi G\,\rho_0^2\,. \tag{D.3}$$

2. Next we obtain the inverse scale height for p_0 and its derivative,

$$\alpha_{\mathrm{p}_0} \;=\; -\frac{\mathrm{p}'_0}{\mathrm{p}_0}\,, \tag{D.4a}$$

$$\alpha'_{\mathrm{p}_0} \;=\; -\frac{\mathrm{p}''_0}{\mathrm{p}_0} \;+\; \alpha_{\mathrm{p}_0}^2\,. \tag{D.4b}$$

3. We have

$$\alpha_{\gamma\mathrm{p}_0} \;=\; 2\alpha_{\mathrm{c}_0} \;+\; \alpha_{\rho_0}\,, \tag{D.5}$$

   which gives us α_γ,

$$\alpha_\gamma \;=\; \alpha_{\mathrm{p}_0} \;-\; \alpha_{\gamma\mathrm{p}_0}\,. \tag{D.6}$$

4. However, after the above step, we run into a problem because if given ρ_0 and ρ'_0 we can get up to $\Phi_0^{(3)}$, however we have the equivalence of the following quantities

$$\mathrm{p}_0^{(3)} \;\Leftrightarrow\; \rho''_0 \;\Leftrightarrow\; \alpha' \tag{D.7}$$

   This can be seen from the ODE of p_0, since n derivatives of p_0 will also require $n-1$ derivatives of ρ_0. In fact all the relations give $\mathrm{p}_0^{(3)} \,+\, \mathrm{p}'_0\,\mathrm{p}''_0$ which is the same as (due to the hydrostatic equilibrium), $\dfrac{1}{\rho_0}\mathrm{p}_0^{(3)} \;-\; \Phi'_0\,\mathrm{p}''_0$.

   Similarly, all the relations give

$$\alpha_{\mathrm{c}_0} \;\Leftrightarrow\; \alpha_{\gamma\mathrm{p}_0} \;\Leftrightarrow\; \alpha_\gamma \tag{D.8}$$

   and on the level of derivative

$$\alpha'_{\mathrm{c}_0} \;\Leftrightarrow\; \alpha'_{\gamma\mathrm{p}_0} \;\Leftrightarrow\; \alpha'_\gamma \tag{D.9}$$

   In particular,

$$-2\alpha_{\mathrm{c}_0} \;+\; \alpha_\gamma \;=\; \ldots \tag{D.10}$$

   This means that we need to assume the a priori computation of α'_{ρ_0} and α'_{c_0}.

D.2 Computational steps for V_ℓ in the interior

Here we compute the background quantities in the same order as in Step 1 – 11 in Subsection 6.3. However, in order to obtain quantities like $r^2 C_{22}(r) q$, $r^2 C_{22}, \tilde{q}$ and V, we start from the original expressions in Proposition 3 (instead of the compact expressions in Proposition 4 and Proposition 7).

Remark 23. *As an alternative to the computational steps 6 to 11 given in Subsection 6.3, we can directly use the scale height function associated with the adiabatic index (i.e., α_γ), and calculate its derivative numerically (e.g., using a finite-differences approximation). We can proceed as follows.*

6. *Compute the scale height function α_γ, and its derivative α_γ'.*
7. *Compute the scale height function for the fluid pressure p_0 and its derivatives*

$$\alpha_{\gamma \mathrm{p}_0} \overset{(6.3)}{=} 2\,\alpha_{\mathrm{c}_0} + \alpha_{\rho_0} \qquad \Rightarrow \qquad \alpha_{\gamma \mathrm{p}_0}' = 2\,\alpha_{\mathrm{c}_0}' + \alpha_{\rho_0}' \tag{D.11a}$$

$$\alpha_{\mathrm{p}_0} \overset{(6.5)}{=} 2\,\alpha_{\mathrm{c}_0} + \alpha_{\rho_0} - \alpha_\gamma \quad \Rightarrow \quad \alpha_{\mathrm{p}_0}' = 2\,\alpha_{\mathrm{c}_0}' + \alpha_{\rho_0}' - \alpha_\gamma' \,, \tag{D.11b}$$

$$\frac{\mathrm{p}_0''}{\mathrm{p}_0} = -\,\alpha_{\mathrm{p}_0}' + \alpha_{\mathrm{p}_0}^2 \,. \tag{D.11c}$$

8. *Compute the derivatives of the background gravitational potential Φ_0,*

$$\Phi_0' \overset{(6.9)}{=} \alpha_{\mathrm{p}_0} \frac{\mathrm{c}_0^2}{\gamma}\,; \tag{D.12a}$$

$$\Phi_0'' \overset{(6.11)}{=} -\frac{\mathrm{p}_0''}{\mathrm{p}_0}\frac{\mathrm{c}_0^2}{\gamma} + \alpha_{\rho_0}\,\Phi_0' \,. \tag{D.12b}$$

△

We can now compute the components of C using (4.35), such that

$$C_{11} = -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\Phi_0''}{\mathrm{c}_0^2} + 2\frac{\alpha_{\gamma \mathrm{p}_0}}{r} + \frac{2}{r^2} - 2\frac{\alpha_{\mathrm{p}_0}}{r\gamma}\,; \tag{D.13a}$$

$$C_{12} = \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}}{r\,\gamma} - \frac{\alpha_{\gamma \mathrm{p}_0}}{r} - \frac{1}{r^2}\right)\,; \tag{D.13b}$$

$$C_{22} = -\frac{\sigma^2}{\mathrm{c}_0^2} + \frac{\ell(\ell+1)}{r^2} \,. \tag{D.13c}$$

The derivatives of C_{22} are given by

$$C_{22}{}' = -\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)' - 2\frac{\ell(\ell+1)}{r^3}\,, \tag{D.14a}$$

$$C_{22}{}'' = -\left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)'' + 6\frac{\ell(\ell+1)}{r^4}\,, \tag{D.14b}$$

and of C_{12} by

$$C_{12}' = \ell(\ell+1)\left(\frac{\alpha_{\mathrm{p}_0}'}{r\,\gamma} + \frac{\alpha_{\mathrm{p}_0}}{r\,\gamma}\alpha_\gamma - \frac{1}{r^2}\frac{\alpha_{\mathrm{p}_0}}{\gamma} - \frac{\alpha_{\gamma \mathrm{p}_0}'}{r} + \frac{\alpha_{\gamma \mathrm{p}_0}}{r^2} + \frac{2}{r^3}\right)\,. \tag{D.15}$$

Next, instead of computing directly $\hat{q}$, $q(r)$, and $\tilde{q}(r)$, which depend on the inverse of the radius:

$$\hat{q}(r) = -1 + \frac{\ell(\ell+1)}{r^2\,C_{22}}\,;$$

$$q(r) = \alpha_{\gamma \mathrm{p}_0} - \frac{2}{r} + \frac{1}{r}\frac{C_{12}}{C_{22}} + \frac{\ell(\ell+1)}{r}\left(\frac{1}{r\,C_{22}}\right)' + \frac{\ell(\ell+1)}{r}\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\,;$$

$$\tilde{q}(r) = C_{11} + \left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{C_{12}}{r\,C_{22}} + \frac{\ell(\ell+1)}{r}\left[\left(\frac{2}{r} - \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\frac{1}{rC_{22}}\right]'\,,$$

we compute a more stable coefficient: $r^2 C_{22}\,\hat{q}$, $r^2 C_{22}\,q$, and $r^2 C_{22}\,\tilde{q}(r)$, given by

$$r^2\,C_{22}\,\hat{q}(r) \;=\; -r^2\,C_{22} \,+\, \ell(\ell+1) \;=\; \frac{\sigma^2}{\mathrm{c}_0^2}r^2\,; \tag{D.16a}$$

$$r^2\,C_{22}\,q(r) \;=\; C_{22}(\alpha_{\gamma\mathrm{p}_0}r^2 \,-\, 2\,r) \,+\, r\,C_{12} \,+\, \ell(\ell+1)\Big(rC_{22}\left(\frac{1}{r\,C_{22}}\right)' \,+\, \frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big)\,; \tag{D.16b}$$

$$\;=\; C_{22}\big(\alpha_{\gamma\mathrm{p}_0}\,r^2 \,-\, 2\,r\big) \,+\, r\,C_{12} \,+\, \ell(\ell+1)\Big(\frac{1}{r} - \frac{C_{22}'}{C_{22}} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\Big)\,; \tag{D.16c}$$

$$r^2\,C_{22}\,\tilde{q}(r) \;=\; r^2C_{22}C_{11} \,+\, \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)' \,+\, \left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{22}\left(\frac{1}{rC_{22}}\right)'\right] \tag{D.16d}$$

$$+\,\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{12} \tag{D.16e}$$

$$= r^2C_{22}C_{11} \,+\, \ell(\ell+1)\left[\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) \,-\, \frac{2}{r^2} \,-\, \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\right] \tag{D.16f}$$

$$+\,\left(\frac{2}{r} \,-\, \frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)rC_{12}\,. \tag{D.16g}$$

Note that in the formulation of $r^2\,C_{22}\,\tilde{q}(r)$ and $r^2\,C_{22}\,q(r)$, we have used

$$\left(\frac{1}{rC_{22}}\right)' \,=\, -\frac{C_{22}\,+\,r\,C_{22}'}{(rC_{22})^2} \quad\Rightarrow\quad r\,C_{22}\left(\frac{1}{r\,C_{22}}\right)' \,=\, -\frac{C_{22}\,+\,r\,C_{22}'}{r\,C_{22}} \,=\, -\frac{1}{r} \,-\, \frac{C_{22}'}{C_{22}}\,.$$

We compute the derivatives,

$$(r^2\,C_{22}\,\hat{q})' \;=\; 2\frac{\sigma^2}{\mathrm{c}_0^2}\,r \,+\, \left(\frac{\sigma^2}{\mathrm{c}_0^2}\right)'r^2\,; \tag{D.17a}$$

$$\begin{aligned}(r^2\,C_{22}\,q)' \;=\;& C_{22}'\big(\alpha_{\gamma\mathrm{p}_0}\,r^2 \,-\, 2\,r\big) \,+\, C_{22}\big(\alpha_{\gamma\mathrm{p}_0}'\,r^2 \,+\, 2\,\alpha_{\gamma\mathrm{p}_0}\,r \,-\, 2\big)\\ &+\, C_{12} \,+\, r\,C_{12}' \,+\, \ell(\ell+1)\Big(-\frac{1}{r^2} \,-\, \frac{C_{22}''}{C_{22}} \,+\, \left(\frac{C_{22}'}{C_{22}}\right)^2 \,-\, \left(\frac{\alpha_{\mathrm{p}_0}}{\gamma}\right)'\Big)\,.\end{aligned} \tag{D.17b}$$

Eventually, we compute the three functions defining the potential:

$$\begin{aligned}\mathfrak{h} \;&=\; -\frac{q}{\hat{q}} \;=\; -\frac{r^2\,C_{22}\,q}{r^2C_{22}\,\hat{q}}\,;\\ \mathfrak{h}' \;&=\; -\frac{(r^2\,C_{22}\,q)'}{r^2C_{22}\,\hat{q}} \,+\, \frac{r^2\,C_{22}\,q}{r^2C_{22}\,\hat{q}}\,\frac{(r^2C_{22}\,\hat{q})'}{r^2C_{22}\,\hat{q}} \;=\; -\frac{(r^2\,C_{22}\,q)'}{r^2C_{22}\,\hat{q}} \,-\, \mathfrak{h}\,\frac{(r^2C_{22}\,\hat{q})'}{r^2C_{22}\,\hat{q}}\,;\\ \mathfrak{g} \;&=\; -\frac{\tilde{q}}{\hat{q}} \;=\; -\frac{r^2\,C_{22}\,\tilde{q}}{r^2\,C_{22}\,\hat{q}}\,.\end{aligned} \tag{D.18}$$

The potential $V_\ell(r)$ is given by

$$V_\ell \;=\; \frac{1}{4}\,\mathfrak{h}^2 \,-\, \frac{1}{2}\,\mathfrak{h}' \,+\, \mathfrak{g}\,. \tag{D.19}$$

D.3 Computational steps for V_ℓ in the atmosphere

Here we compute the background quantities in the same order as in Step 1 of Subsection 6.4. However, in order to obtain quantities like $r^2C_{22}(r)q$, $r^2C_{22}, \tilde{q}$ and V, we start from the original expressions in Proposition 3 (instead of the compact expressions in Proposition 5 and Proposition 8).

We have the following steps for the computation of the potential in the atmosphere.

1. This step remains as Step 1 of Subsection 6.4.

2. Compute the components of the matrix C from (4.39) (note that they are functions of r),

$$C_{11} = -\frac{\sigma^2}{c_0^2} + \frac{2}{r}\left(\alpha_{\rho_0} - \frac{\alpha_{\rho_0}}{\gamma}\right) + \frac{2}{r^2} + \frac{\Phi_0''}{c_0^2}; \tag{D.20a}$$

$$C_{12} = \ell(\ell+1)\left(-\frac{\alpha_{\rho_0}}{r} + \frac{\alpha_{\rho_0}}{r\,\gamma} - \frac{1}{r^2}\right); \tag{D.20b}$$

$$C_{22} = -\frac{\sigma^2}{c_0^2} - \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r} + \frac{\ell(\ell+1)}{r^2} + \frac{\Phi_0'}{c_0^2}\frac{1}{r}. \tag{D.20c}$$

3. Compute the derivatives of C_{22}

$$\begin{aligned}
C_{22}' &= \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2} - 2\frac{\ell(\ell+1)}{r^3} + \frac{1}{c_0^2}\left(\frac{\Phi_0''}{r} - \frac{\Phi_0'}{r^2}\right);\\
&\overset{(4.16)}{=} \frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^2} - 2\frac{\ell(\ell+1)}{r^3} + \frac{1}{c_0^2}\frac{1}{r^2}\left(4\pi G\,\rho_0\,r - 3\Phi_0'\right),\\
C_{22}'' &= -2\frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^3} + 6\frac{\ell(\ell+1)}{r^4} - \frac{2}{c_0^2}\frac{1}{r^3}\left(4\pi G\,\rho_0\,r - 3\Phi_0'\right)\\
&\qquad + \frac{1}{c_0^2}\frac{1}{r^2}\left(4\pi G\,\rho_0 + 4\pi G\,\rho_0' r - 12\pi G\rho_0 + \frac{6}{r}\Phi_0'\right)\\
&= -2\frac{\alpha_{\rho_0}}{\gamma}\frac{1}{r^3} + 6\frac{\ell(\ell+1)}{r^4} - \frac{4}{c_0^2}\frac{1}{r^3}\left(4\pi G\,\rho_0\,r - 3\Phi_0'\right) - 4\,\pi\,G\,\frac{\rho_0\,\alpha_{\rho_0}}{c_0^2\,r}.
\end{aligned} \tag{D.21}$$

and of C_{12},

$$C_{12}' = \ell(\ell+1)\left(\frac{\alpha_{\rho_0}}{r^2} - \frac{\alpha_{\rho_0}}{r^2\,\gamma} + \frac{2}{r^3}\right). \tag{D.22}$$

4. Similarly to the computation in the interior, instead of computing $\hat{q}$, $q(r)$, and $\tilde{q}(r)$, we use $r^2 C_{22}\,\hat{q}$, $r^2 C_{22}\,q$, and $r^2 C_{22}\,\tilde{q}(r)$, such that

$$r^2\,C_{22}\,\hat{q}(r) = -r^2\,C_{22} + \ell(\ell+1) = \frac{\sigma^2}{c_0^2}r^2 + \frac{\alpha_{\rho_0}}{\gamma}r - \frac{\Phi_0'}{c_0^2}r; \tag{D.23a}$$

$$r^2\,C_{22}\,q(r) = C_{22}(\alpha_{\rho_0}\,r^2 - 2\,r) + r\,C_{12} + \ell(\ell+1)\left(rC_{22}\left(\frac{1}{r\,C_{22}}\right)' + \frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right); \tag{D.23b}$$

$$= C_{22}(\alpha_{\rho_0}\,r^2 - 2\,r) + r\,C_{12} + \ell(\ell+1)\left(\frac{1}{r} - \frac{C_{22}'}{C_{22}} - \frac{\alpha_{\rho_0}}{\gamma}\right); \tag{D.23c}$$

$$r^2\,C_{22}\,\tilde{q}(r) = r^2C_{22}C_{11} + \ell(\ell+1)\Big[\big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)' \tag{D.23d}$$

$$+ \big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)rC_{22}\big(\frac{1}{rC_{22}}\big)'\Big] + \big(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\big)rC_{12} \tag{D.23e}$$

$$= r^2C_{22}C_{11} + \ell(\ell+1)\left[\left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)\left(-\frac{1}{r} - \frac{C_{22}'}{C_{22}}\right) - \frac{2}{r^2}\right] \tag{D.23f}$$

$$+ \left(\frac{2}{r} - \frac{\alpha_{\rho_0}}{\gamma}\right)rC_{12}. \tag{D.23g}$$

5. Compute the derivatives,

$$\begin{aligned}
(r^2\,C_{22}\,\hat{q})' &= 2\frac{\sigma^2}{c_0^2}r + \frac{\alpha_{\rho_0}}{\gamma} - \frac{\Phi_0'}{c_0^2} - \frac{\Phi_0''}{c_0^2}r;\\
(r^2\,C_{22}\,q)' &= C_{22}'\,(\alpha_{\rho_0}\,r^2 - 2\,r) + C_{22}\,(2\,\alpha_{\rho_0}\,r - 2) + C_{12} + r\,C_{12}'\\
&\qquad + \ell(\ell+1)\left(-\frac{1}{r^2} - \frac{C_{22}''}{C_{22}} + \left(\frac{C_{22}'}{C_{22}}\right)^2\right).
\end{aligned} \tag{D.24}$$

6. We compute the three functions,

$$\begin{aligned}
\mathfrak{h} &= -\frac{q}{\hat{q}} = -\frac{r^2\, C_{22}\, q}{r^2 C_{22}\, \hat{q}}\,;\\
\mathfrak{h}' &= -\frac{(r^2\, C_{22}\, q)'}{r^2 C_{22}\, \hat{q}} + \frac{r^2\, C_{22}\, q}{r^2 C_{22}\, \hat{q}}\,\frac{(r^2 C_{22}\, \hat{q})'}{r^2 C_{22}\, \hat{q}} = -\frac{(r^2\, C_{22}\, q)'}{r^2 C_{22}\, \hat{q}} - \mathfrak{h}\,\frac{(r^2 C_{22}\, \hat{q})'}{r^2 C_{22}\, \hat{q}}\,;\\
\mathfrak{g} &= -\frac{\tilde{q}}{\hat{q}} = -\frac{r^2\, C_{22}\, \tilde{q}}{r^2\, C_{22}\, \hat{q}}\,.
\end{aligned} \tag{D.25}$$

7. Eventually, the potential in the atmosphere is given by

$$V_\ell = \frac{1}{4}\,\mathfrak{h}^2 - \frac{1}{2}\,\mathfrak{h}' + \mathfrak{g}\,. \tag{D.26}$$

E Square roots of potential

Recall that

$$-V_\ell(r) = Q(r) + \varepsilon_V(r)\,, \qquad \varepsilon_V = -V_\ell(r) - Q(r) = \mathsf{O}(r^{-3})\,,$$

where Q consists of the first three summands in the asymptotic expansion of $(-V_\ell)$

$$\begin{aligned}
Q(r) &:= \mathsf{k}^2 + \frac{\alpha_{\rm ad}}{r} - \frac{\mu_\ell^2 - \frac{1}{4}}{r^2}\\
&= \mathsf{k}^2 + \frac{\alpha_{\rm ad}}{r} - \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{\alpha^2}{k_0^2}\,\frac{\gamma-1}{\gamma^2}\right)\frac{1}{r^2}\,.
\end{aligned} \tag{E.1}$$

We recall the definitions of the wavenumbers,

$$\sigma^2 = \omega^2 + 2\mathrm{i}\,\Gamma_a\,\omega = \omega^2\big(1 + 2\mathrm{i}\,\frac{\Gamma_a}{\omega}\big)\,; \tag{E.2a}$$

$$\mathsf{k}^2 = k_0^2 - \frac{\alpha^2}{4} = \frac{\omega^2}{\mathrm{c}_0^2} + 2\mathrm{i}\,\frac{\Gamma_a\,\omega}{\mathrm{c}_0^2} - \frac{\alpha^2}{4\mathrm{c}_0^2} \qquad \Rightarrow \qquad \mathrm{Im}\,\mathsf{k}^2 = \mathrm{Im}\;k_0^2 = 2\frac{\Gamma_a}{\omega}\left(\frac{\omega}{\mathrm{c}_0}\right)^2; \tag{E.2b}$$

$$\frac{1}{k_0^2} = \frac{1}{|k_0|^4}\left(\frac{\omega^2}{\mathrm{c}_0^2} - 2\mathrm{i}\frac{\Gamma_a\,\omega}{\mathrm{c}_0^2}\right) \qquad \Rightarrow \qquad \mathrm{Im}\;\frac{1}{k_0^2} = -2\,\frac{1}{|\mathsf{k}_0|^4}\,\frac{\Gamma_a}{\omega}\left(\frac{\omega}{\mathrm{c}_0}\right)^2. \tag{E.2c}$$

And we have defined

$$\sigma := \sqrt{\sigma^2 + \mathrm{i}\,\omega\,\frac{2\,\Gamma}{\mathrm{c}_0^2}}\,, \qquad k_0 := \sqrt{k_0^2} = \frac{\sigma}{\mathrm{c}_0}\,. \tag{E.3}$$

We denote by $\sqrt{\ }$ the square root branch that uses Argument branch $[0, 2\pi)$ while $()^{1/2}$ uses Argument branch $(-\pi, \pi]$. With $\Gamma_0 > 0$, we always have

$$\boxed{\sqrt{\mathsf{k}^2} = (\mathsf{k}^2)^{1/2}\,, \qquad \mathrm{Im}\,\mathsf{k} > 0\,.} \tag{E.4}$$

Square root of Q We need to investigate the sign of its imaginary part. Since

$$\mu_\ell^2 = \frac{1}{4} + \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{\alpha^2\,(\gamma-1)}{\gamma^2}\,\frac{1}{\mathsf{k}_0^2}\right),$$

and replacing $|\mathsf{k}_0^2|$ by

$$|\mathsf{k}_0^2| = \frac{|\sigma^2|}{\mathrm{c}_0^2} = \frac{\omega^2}{\mathrm{c}_0^2}\sqrt{1 + \frac{4\,\Gamma_a^2}{\omega^2}}\,,$$

we obtain the following expressions for the real and imaginary part of μ_ℓ^2,

$$\begin{aligned}
\mathrm{Re}\,\mu_\ell^2 &= \frac{1}{4} + \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{\alpha^2(\gamma-1)}{\gamma^2}\left(\mathrm{Re}\,\frac{1}{k_0^2}\right)\right)\\
&= \frac{1}{4} + \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{\alpha^2\,(\gamma-1)}{\gamma^2}\,\frac{\omega^2}{|\mathsf{k}_0|^4\,\mathrm{c}_0^2}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}\operatorname{Re}\mu_\ell^2 &= \frac{1}{4} + \ell(\ell+1)\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{1}{\left(\frac{\omega}{c_0}\right)^2}\frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\right), \\ &= \left(\ell+\frac{1}{2}\right)^2 + 2 - \frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{\left(\frac{\omega}{c_0}\right)^2}.\end{aligned} \tag{E.5}$$

Similarly, for the imaginary part, we have

$$\begin{aligned}\operatorname{Im}\mu_\ell^2 &= -\ell(\ell+1)\frac{\alpha^2\,(\gamma-1)}{\gamma^2}\left(\operatorname{Im}\frac{1}{\mathsf{k}_0^2}\right) \quad = -\ell(\ell+1)\,\frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{(-2)\,\Gamma_a\,\omega}{|k_0|^4\,c_0^2} \\ &= 2\frac{\Gamma_a\,\omega}{c_0^2}\frac{\alpha^2}{|k_0|^4}\frac{\gamma-1}{\gamma^2}\ell(\ell+1) \quad = 2\frac{\Gamma_a}{\omega}\left(\frac{\omega}{c_0}\right)^2\frac{\alpha^2}{(\frac{\omega}{c_0})^2}\frac{(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{(\frac{\omega}{c_0})^2}.\end{aligned}$$

This leads to,

$$\operatorname{Im}\mu_\ell^2 = 2\,\frac{\Gamma_a}{\omega}\,\frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{(\frac{\omega}{c_0})^2}. \tag{E.6}$$

From these calculations, we have

$$\begin{aligned}\operatorname{Im}Q &= \operatorname{Im}\mathsf{k}^2 - \frac{\operatorname{Im}\mu_\ell^2}{r^2} \quad = 2\,\frac{\Gamma_a}{\omega}\left(\left(\frac{\omega}{c_0}\right)^2 - \frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{(\frac{\omega}{c_0}\,r)^2}\right); \\ \operatorname{Re}Q &= \operatorname{Re}\mathsf{k}^2 + \frac{\alpha_{\mathrm{ad}}}{r} - \frac{\operatorname{Re}\mu_\ell^2 - \frac{1}{4}}{r^2} \\ &= \left(\frac{\omega^2}{c_0^2} - \frac{\alpha^2}{4}\right) + \frac{\alpha_{\mathrm{ad}}}{r} - \frac{\ell(\ell+1)}{r^2}\left(\frac{2}{\ell(\ell+1)} + 1 - \frac{1}{(\frac{\omega}{c_0})^2}\frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\right) \\ &= \left(\frac{\omega^2}{c_0^2} - \frac{\alpha^2}{4}\right) + \frac{\alpha_{\mathrm{ad}}}{r} - \frac{1}{r^2}\left(\ell(\ell+1) + 2 - \frac{\alpha^2\,(\gamma-1)}{\gamma^2}\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{(\frac{\omega}{c_0})^2}\right).\end{aligned}$$

Then, we use the physical assumption (2.1), that

$$\boxed{1 < \gamma < 2\,.}$$

In addition, we note that if ω, r, and ℓ satisfy

$$\boxed{1 - \left(\frac{\alpha}{\frac{\omega}{c_0}\,\gamma}\right)^2(\gamma-1)\,\frac{1}{1+4(\frac{\Gamma_a}{\omega})^2}\frac{\ell(\ell+1)}{(\frac{\omega}{c_0}\,r)^2} > 0\,, \qquad \ell > 0}, \tag{E.7}$$

then

$$\operatorname{Im}Q > 0\,. \tag{E.8}$$

Remark 24. *Under the hypothesis (2.1) and (E.7), we also have*

$$\operatorname{Re}\mu_\ell^2 > \tfrac{1}{4}\,, \qquad \operatorname{Im}\mu_\ell^2 > 0\,. \tag{E.9}$$

△

F Definition of the solar models using splines

In the model S from [12], the physical parameters (density and velocity) for the Sun are given point-wise, that is, a list of (spherical) positions associated with the value of the parameters. In order to work with their derivatives, which are required in the vectorial case through the scale heights, we extract a cubic spline representation from the given point-wise representation. Namely, we proceed as follows.

1. We start from a coarse partition to generate a first cubic spline representation.
2. We compute the difference between the original values and the spline representation.
3. The interval with the maximal difference is refined to increase the number of splines and reduce the difference with the original model.
4. We repeat until the error is less than a selected threshold.

Consequently, we follow an iterative refinement of a coarse interval and the resulting basis of splines is unstructured. This allows us to explicitly form the parameters and their derivatives. For instance, we give in Table 2 the velocity model represented with cubic splines, which is pictured in Figure 1a.

Table 2: Coefficients to define the solar velocity model using splines, generated from the model `S`. On each interval $[x_1, x_2]$, the model is given by the function $a(x-x_1)^3 + b(x-x_1)^2 + c(x-x_1) + d$.

Interval start	Interval end	Spline coeff a	Spline coeff b	Spline coeff c	Spline coeff d
0	0.0001248	$8.5965113 \cdot 10^{6}$	$5.6334513 \cdot 10^{5}$	$9.2473625 \cdot 10^{4}$	$5.0356619 \cdot 10^{5}$
0.0001248	0.0188748	$8.5965113 \cdot 10^{6}$	$5.6656237 \cdot 10^{5}$	$9.2614581 \cdot 10^{4}$	$5.0357774 \cdot 10^{5}$
0.0188748	0.0376248	$-2.2302045 \cdot 10^{7}$	$1.0501161 \cdot 10^{6}$	$1.229273 \cdot 10^{5}$	$5.0557011 \cdot 10^{5}$
0.0376248	0.0563748	$-4.046857 \cdot 10^{7}$	$-2.043739 \cdot 10^{5}$	$1.3878497 \cdot 10^{5}$	$5.0809717 \cdot 10^{5}$
0.0563748	0.0751248	$-2.3953166 \cdot 10^{7}$	$-2.480731 \cdot 10^{6}$	$8.8439254 \cdot 10^{4}$	$5.1036078 \cdot 10^{5}$
0.0751248	0.0938748	$-6.4827862 \cdot 10^{6}$	$-3.8280965 \cdot 10^{6}$	$-2.9851263 \cdot 10^{4}$	$5.1098899 \cdot 10^{5}$
0.0938748	0.1126248	$6.1274876 \cdot 10^{6}$	$-4.1927533 \cdot 10^{6}$	$-1.802422 \cdot 10^{5}$	$5.0904073 \cdot 10^{5}$
0.1126248	0.1313748	$1.2578428 \cdot 10^{7}$	$-3.8480821 \cdot 10^{6}$	$-3.3100786 \cdot 10^{5}$	$5.0422756 \cdot 10^{5}$
0.1313748	0.1501248	$1.4673379 \cdot 10^{7}$	$-3.1405455 \cdot 10^{6}$	$-4.6204463 \cdot 10^{5}$	$4.9675124 \cdot 10^{5}$
0.1501248	0.1688748	$1.549011 \cdot 10^{7}$	$-2.315168 \cdot 10^{6}$	$-5.6433926 \cdot 10^{5}$	$4.8708053 \cdot 10^{5}$
0.1688748	0.1876248	$1.4359805 \cdot 10^{7}$	$-1.4438493 \cdot 10^{6}$	$-6.3482083 \cdot 10^{5}$	$4.7578735 \cdot 10^{5}$
0.1876248	0.2063748	$1.0138205 \cdot 10^{7}$	$-6.3611027 \cdot 10^{5}$	$-6.7382007 \cdot 10^{5}$	$4.6347151 \cdot 10^{5}$
0.2063748	0.2251248	$6.1668004 \cdot 10^{6}$	$-6.5836217 \cdot 10^{4}$	$-6.8698157 \cdot 10^{5}$	$4.5068058 \cdot 10^{5}$
0.2251248	0.2438748	$4.5230315 \cdot 10^{6}$	$2.8104631 \cdot 10^{5}$	$-6.8294638 \cdot 10^{5}$	$4.3781718 \cdot 10^{5}$
0.2438748	0.2626248	$3.296957 \cdot 10^{6}$	$5.3546683 \cdot 10^{5}$	$-6.6763676 \cdot 10^{5}$	$4.2514056 \cdot 10^{5}$
0.2626248	0.2813748	$3.0122119 \cdot 10^{6}$	$7.2092066 \cdot 10^{5}$	$-6.4407949 \cdot 10^{5}$	$4.1283235 \cdot 10^{5}$
0.2813748	0.3001248	$-1.2519474 \cdot 10^{6}$	$8.9035758 \cdot 10^{5}$	$-6.1386802 \cdot 10^{5}$	$4.0102916 \cdot 10^{5}$
0.3001248	0.3188748	$-1.4725971 \cdot 10^{6}$	$8.1993554 \cdot 10^{5}$	$-5.8180003 \cdot 10^{5}$	$3.898239 \cdot 10^{5}$
0.3188748	0.3376248	$-5.5787909 \cdot 10^{5}$	$7.3710195 \cdot 10^{5}$	$-5.5260558 \cdot 10^{5}$	$3.791937 \cdot 10^{5}$
0.3376248	0.3563748	$-4.0520594 \cdot 10^{5}$	$7.0572126 \cdot 10^{5}$	$-5.2555264 \cdot 10^{5}$	$3.6908781 \cdot 10^{5}$
0.3563748	0.3751248	$-1.6387087 \cdot 10^{6}$	$6.8292842 \cdot 10^{5}$	$-4.9951546 \cdot 10^{5}$	$3.5947913 \cdot 10^{5}$
0.3751248	0.3938748	$-6.1433805 \cdot 10^{5}$	$5.9075106 \cdot 10^{5}$	$-4.7563397 \cdot 10^{5}$	$3.503425 \cdot 10^{5}$
0.3938748	0.4126248	$-1.3627564 \cdot 10^{5}$	$5.5619454 \cdot 10^{5}$	$-4.5412874 \cdot 10^{5}$	$3.41628 \cdot 10^{5}$
0.4126248	0.4313748	$-1.5380114 \cdot 10^{6}$	$5.4852904 \cdot 10^{5}$	$-4.3341517 \cdot 10^{5}$	$3.3330773 \cdot 10^{5}$
0.4313748	0.4501248	$-2.4515616 \cdot 10^{5}$	$4.620159 \cdot 10^{5}$	$-4.1446745 \cdot 10^{5}$	$3.253639 \cdot 10^{5}$
0.4501248	0.4688748	$-9.9023438 \cdot 10^{5}$	$4.4822586 \cdot 10^{5}$	$-3.9740042 \cdot 10^{5}$	$3.1775345 \cdot 10^{5}$
0.4688748	0.4876248	$-8.3097292 \cdot 10^{5}$	$3.9252518 \cdot 10^{5}$	$-3.8163634 \cdot 10^{5}$	$3.1045324 \cdot 10^{5}$
0.4876248	0.5063748	$-9.876261 \cdot 10^{5}$	$3.4578295 \cdot 10^{5}$	$-3.6779306 \cdot 10^{5}$	$3.0343008 \cdot 10^{5}$
0.5063748	0.5251248	$-1.5564943 \cdot 10^{5}$	$2.9022898 \cdot 10^{5}$	$-3.5586784 \cdot 10^{5}$	$2.9664901 \cdot 10^{5}$
0.5251248	0.5438748	$-2.2539198 \cdot 10^{6}$	$2.814737 \cdot 10^{5}$	$-3.4514841 \cdot 10^{5}$	$2.900775 \cdot 10^{5}$
0.5438748	0.5626248	$7.3921187 \cdot 10^{5}$	$1.5469071 \cdot 10^{5}$	$-3.3697033 \cdot 10^{5}$	$2.8369006 \cdot 10^{5}$
0.5626248	0.5813748	$-4.0477424 \cdot 10^{6}$	$1.9627138 \cdot 10^{5}$	$-3.3038979 \cdot 10^{5}$	$2.7743113 \cdot 10^{5}$
0.5813748	0.6001248	$2.0580451 \cdot 10^{6}$	$-3.1414129 \cdot 10^{4}$	$-3.2729872 \cdot 10^{5}$	$2.7127864 \cdot 10^{5}$
0.6001248	0.6188748	$-7.0544998 \cdot 10^{6}$	$8.4350905 \cdot 10^{4}$	$-3.2630615 \cdot 10^{5}$	$2.6514431 \cdot 10^{5}$
0.6188748	0.6376248	$4.8225952 \cdot 10^{6}$	$-3.1246471 \cdot 10^{5}$	$-3.3058329 \cdot 10^{5}$	$2.5900922 \cdot 10^{5}$
0.6376248	0.6563748	$-1.0507944 \cdot 10^{7}$	$-4.1193729 \cdot 10^{4}$	$-3.3721438 \cdot 10^{5}$	$2.5273272 \cdot 10^{5}$
0.6563748	0.6751248	$1.8470242 \cdot 10^{6}$	$-6.3226556 \cdot 10^{5}$	$-3.4984174 \cdot 10^{5}$	$2.463262 \cdot 10^{5}$
0.6751248	0.6843403	$2.3986315 \cdot 10^{5}$	$-5.2837045 \cdot 10^{5}$	$-3.7160367 \cdot 10^{5}$	$2.3955657 \cdot 10^{5}$
0.6843403	0.6938748	$-5.3840623 \cdot 10^{7}$	$-5.2173907 \cdot 10^{5}$	$-3.8128095 \cdot 10^{5}$	$2.3608737 \cdot 10^{5}$
0.6938748	0.6968403	$-8.835186 \cdot 10^{7}$	$-2.0617693 \cdot 10^{6}$	$-4.0591341 \cdot 10^{5}$	$2.3235795 \cdot 10^{5}$
0.6968403	0.7093403	$-8.7080603 \cdot 10^{6}$	$-2.8477917 \cdot 10^{6}$	$-4.2047272 \cdot 10^{5}$	$2.3113378 \cdot 10^{5}$
0.7093403	0.7126248	$7.3387194 \cdot 10^{7}$	$-3.1743439 \cdot 10^{6}$	$-4.9574941 \cdot 10^{5}$	$2.2541589 \cdot 10^{5}$
0.7126248	0.7218403	$5.9056073 \cdot 10^{7}$	$-2.4512232 \cdot 10^{6}$	$-5.1422659 \cdot 10^{5}$	$2.2375596 \cdot 10^{5}$
0.7218403	0.7313748	$4.6738917 \cdot 10^{7}$	$-8.1852948 \cdot 10^{5}$	$-5.4435899 \cdot 10^{5}$	$2.1885515 \cdot 10^{5}$
0.7313748	0.7501248	$-1.4128445 \cdot 10^{7}$	$5.1836715 \cdot 10^{5}$	$-5.4722089 \cdot 10^{5}$	$2.1363106 \cdot 10^{5}$
0.7501248	0.7688748	$5.9224776 \cdot 10^{6}$	$-2.7635787 \cdot 10^{5}$	$-5.4268322 \cdot 10^{5}$	$2.0345978 \cdot 10^{5}$
0.7688748	0.7876248	$-5.8563137 \cdot 10^{6}$	$5.6781493 \cdot 10^{4}$	$-5.4680027 \cdot 10^{5}$	$1.9322635 \cdot 10^{5}$
0.7876248	0.8063748	$1.4925678 \cdot 10^{6}$	$-2.7263615 \cdot 10^{5}$	$-5.5084755 \cdot 10^{5}$	$1.829552 \cdot 10^{5}$
0.8063748	0.8251248	$-2.4281431 \cdot 10^{6}$	$-1.8867921 \cdot 10^{5}$	$-5.5949721 \cdot 10^{5}$	$1.725408 \cdot 10^{5}$
0.8251248	0.8438748	$-3.3351645 \cdot 10^{6}$	$-3.2526226 \cdot 10^{5}$	$-5.6913361 \cdot 10^{5}$	$1.6196789 \cdot 10^{5}$
0.8438748	0.8626248	$-2.3718634 \cdot 10^{6}$	$-5.1286527 \cdot 10^{5}$	$-5.8484851 \cdot 10^{5}$	$1.511603 \cdot 10^{5}$

0.8626248	0.8813748	$-4.303895 \cdot 10^{6}$	$-6.4628258 \cdot 10^{5}$	$-6.0658253 \cdot 10^{5}$	$1.3999845 \cdot 10^{5}$
0.8813748	0.9001248	$-6.1965032 \cdot 10^{6}$	$-8.8837668 \cdot 10^{5}$	$-6.3535739 \cdot 10^{5}$	$1.2836945 \cdot 10^{5}$
0.9001248	0.9188748	$-1.0805591 \cdot 10^{7}$	$-1.23693 \cdot 10^{6}$	$-6.7520689 \cdot 10^{5}$	$1.1610333 \cdot 10^{5}$
0.9188748	0.9268075	$-7.8233197 \cdot 10^{6}$	$-1.8447445 \cdot 10^{6}$	$-7.3298828 \cdot 10^{5}$	$1.0293712 \cdot 10^{5}$
0.9268075	0.9376248	$-4.0647786 \cdot 10^{7}$	$-2.0309258 \cdot 10^{6}$	$-7.6373301 \cdot 10^{5}$	$9.7002513 \cdot 10^{4}$
0.9376248	0.9393075	$1.1189644 \cdot 10^{8}$	$-3.3500176 \cdot 10^{6}$	$-8.2194002 \cdot 10^{5}$	$8.8451927 \cdot 10^{4}$
0.9393075	0.9518075	$-5.6206069 \cdot 10^{7}$	$-2.7851364 \cdot 10^{6}$	$-8.3226395 \cdot 10^{5}$	$8.7059855 \cdot 10^{4}$
0.9518075	0.9533675	$-2.712999 \cdot 10^{8}$	$-4.892864 \cdot 10^{6}$	$-9.2823895 \cdot 10^{5}$	$7.6111601 \cdot 10^{4}$
0.9533675	0.9563748	$2.0022128 \cdot 10^{8}$	$-6.1625475 \cdot 10^{6}$	$-9.4548539 \cdot 10^{5}$	$7.4650611 \cdot 10^{4}$
0.9563748	0.9643075	$-2.5846327 \cdot 10^{8}$	$-4.3562012 \cdot 10^{6}$	$-9.771179 \cdot 10^{5}$	$7.1757014 \cdot 10^{4}$
0.9643075	0.9658675	$2.91817 \cdot 10^{8}$	$-1.0507175 \cdot 10^{7}$	$-1.0950253 \cdot 10^{6}$	$6.3602628 \cdot 10^{4}$
0.9658675	0.968422	$3.8830861 \cdot 10^{8}$	$-9.1414712 \cdot 10^{6}$	$-1.1256772 \cdot 10^{6}$	$6.1869926 \cdot 10^{4}$
0.968422	0.9718033	$-6.8085932 \cdot 10^{8}$	$-6.1656682 \cdot 10^{6}$	$-1.1647793 \cdot 10^{6}$	$5.8941204 \cdot 10^{4}$
0.9718033	0.9719283	$7.9528012 \cdot 10^{8}$	$-1.3072135 \cdot 10^{7}$	$-1.2298271 \cdot 10^{6}$	$5.4905983 \cdot 10^{4}$
0.9719283	0.9720533	$-1.0405377 \cdot 10^{9}$	$-1.2773905 \cdot 10^{7}$	$-1.2330579 \cdot 10^{6}$	$5.4752052 \cdot 10^{4}$
0.9720533	0.9721783	$2.3273007 \cdot 10^{9}$	$-1.3164107 \cdot 10^{7}$	$-1.2363002 \cdot 10^{6}$	$5.4597718 \cdot 10^{4}$
0.9721783	0.9723033	$-5.3376184 \cdot 10^{9}$	$-1.2291369 \cdot 10^{7}$	$-1.2394821 \cdot 10^{6}$	$5.4442979 \cdot 10^{4}$
0.9723033	0.9751248	$1.6866833 \cdot 10^{9}$	$-1.4292976 \cdot 10^{7}$	$-1.2428051 \cdot 10^{6}$	$5.4287841 \cdot 10^{4}$
0.9751248	0.9783675	$-1.6216581 \cdot 10^{8}$	$-1.6044623 \cdot 10^{4}$	$-1.283178 \cdot 10^{6}$	$5.0705368 \cdot 10^{4}$
0.9783675	0.979321	$4.1085498 \cdot 10^{8}$	$-1.5936341 \cdot 10^{6}$	$-1.2883978 \cdot 10^{6}$	$4.6538644 \cdot 10^{4}$
0.979321	0.980922	$3.5197506 \cdot 10^{8}$	$-4.1838346 \cdot 10^{5}$	$-1.2903163 \cdot 10^{6}$	$4.5309064 \cdot 10^{4}$
0.980922	0.9816958	$-5.1998225 \cdot 10^{9}$	$1.2721528 \cdot 10^{6}$	$-1.2889494 \cdot 10^{6}$	$4.324364 \cdot 10^{4}$
0.9816958	0.9826908	$-4.5353758 \cdot 10^{9}$	$-1.0797935 \cdot 10^{7}$	$-1.29632 \cdot 10^{6}$	$4.2244668 \cdot 10^{4}$
0.9826908	0.9851908	$-6.1873693 \cdot 10^{9}$	$-2.4336032 \cdot 10^{7}$	$-1.3312783 \cdot 10^{6}$	$4.0939672 \cdot 10^{4}$
0.9851908	0.98675	$-2.0752238 \cdot 10^{9}$	$-7.0741302 \cdot 10^{7}$	$-1.5689716 \cdot 10^{6}$	$3.7362698 \cdot 10^{4}$
0.98675	0.986821	$6.0252292 \cdot 10^{10}$	$-8.044868 \cdot 10^{7}$	$-1.8047146 \cdot 10^{6}$	$3.4736422 \cdot 10^{4}$
0.986821	0.9868215	$-9.9730508 \cdot 10^{12}$	$-6.7614942 \cdot 10^{7}$	$-1.8152271 \cdot 10^{6}$	$3.4607903 \cdot 10^{4}$
0.9868215	0.9870715	$5.7302679 \cdot 10^{9}$	$-8.2574518 \cdot 10^{7}$	$-1.8153022 \cdot 10^{6}$	$3.4606995 \cdot 10^{4}$
0.9870715	0.9873215	$-1.9482435 \cdot 10^{8}$	$-7.8276817 \cdot 10^{7}$	$-1.855515 \cdot 10^{6}$	$3.4148098 \cdot 10^{4}$
0.9873215	0.9875715	$5.7448459 \cdot 10^{9}$	$-7.8422935 \cdot 10^{7}$	$-1.89469 \cdot 10^{6}$	$3.3679324 \cdot 10^{4}$
0.9875715	0.9876908	$8.563036 \cdot 10^{9}$	$-7.4114301 \cdot 10^{7}$	$-1.9328243 \cdot 10^{6}$	$3.320084 \cdot 10^{4}$
0.9876908	0.9878215	$4.5710986 \cdot 10^{9}$	$-7.1050875 \cdot 10^{7}$	$-1.9501352 \cdot 10^{6}$	$3.2969311 \cdot 10^{4}$
0.9878215	0.988	$2.1450425 \cdot 10^{10}$	$-6.9257861 \cdot 10^{7}$	$-1.9684806 \cdot 10^{6}$	$3.2713127 \cdot 10^{4}$
0.988	0.98925	$4.589262 \cdot 10^{9}$	$-5.7771159 \cdot 10^{7}$	$-1.9911553 \cdot 10^{6}$	$3.2359668 \cdot 10^{4}$
0.98925	0.9901908	$2.7195561 \cdot 10^{10}$	$-4.0561426 \cdot 10^{7}$	$-2.114071 \cdot 10^{6}$	$2.978942 \cdot 10^{4}$
0.9901908	0.9905	$1.0270578 \cdot 10^{10}$	$3.6191247 \cdot 10^{7}$	$-2.1181822 \cdot 10^{6}$	$2.7787353 \cdot 10^{4}$
0.9905	0.9908675	$-5.3284279 \cdot 10^{10}$	$4.5719776 \cdot 10^{7}$	$-2.0928513 \cdot 10^{6}$	$2.713607 \cdot 10^{4}$
0.9908675	0.991696	$1.4845111 \cdot 10^{10}$	$-1.3026141 \cdot 10^{7}$	$-2.0808363 \cdot 10^{6}$	$2.6370477 \cdot 10^{4}$
0.991696	0.991721	$4.0145258 \cdot 10^{11}$	$2.3871381 \cdot 10^{7}$	$-2.0718511 \cdot 10^{6}$	$2.4646005 \cdot 10^{4}$
0.991721	0.991746	$3.1072857 \cdot 10^{11}$	$5.3980325 \cdot 10^{7}$	$-2.0699048 \cdot 10^{6}$	$2.459423 \cdot 10^{4}$
0.991746	0.99175	$-3.4007032 \cdot 10^{12}$	$7.7284968 \cdot 10^{7}$	$-2.0666231 \cdot 10^{6}$	$2.4542521 \cdot 10^{4}$
0.99175	0.991771	$5.1690929 \cdot 10^{11}$	$3.6476529 \cdot 10^{7}$	$-2.0661681 \cdot 10^{6}$	$2.4534256 \cdot 10^{4}$
0.991771	0.991796	$-9.5776893 \cdot 10^{10}$	$6.9041814 \cdot 10^{7}$	$-2.0639522 \cdot 10^{6}$	$2.4490887 \cdot 10^{4}$
0.991796	0.991821	$1.7599631 \cdot 10^{11}$	$6.1858547 \cdot 10^{7}$	$-2.0606797 \cdot 10^{6}$	$2.443933 \cdot 10^{4}$
0.991821	0.991846	$-4.6468553 \cdot 10^{10}$	$7.5058271 \cdot 10^{7}$	$-2.0572568 \cdot 10^{6}$	$2.4387854 \cdot 10^{4}$
0.991846	0.991871	$3.6977122 \cdot 10^{10}$	$7.1573129 \cdot 10^{7}$	$-2.053591 \cdot 10^{6}$	$2.4336469 \cdot 10^{4}$
0.991871	0.991896	$1.4792608 \cdot 10^{11}$	$7.4346414 \cdot 10^{7}$	$-2.049943 \cdot 10^{6}$	$2.4285175 \cdot 10^{4}$
0.991896	0.991921	$-1.5505596 \cdot 10^{11}$	$8.544087 \cdot 10^{7}$	$-2.0459483 \cdot 10^{6}$	$2.4233975 \cdot 10^{4}$
0.991921	0.991946	$4.8140778 \cdot 10^{11}$	$7.3811673 \cdot 10^{7}$	$-2.041967 \cdot 10^{6}$	$2.4182877 \cdot 10^{4}$
0.991946	0.9926908	$-4.2461762 \cdot 10^{10}$	$1.0991726 \cdot 10^{8}$	$-2.0373738 \cdot 10^{6}$	$2.4131882 \cdot 10^{4}$
0.9926908	0.993422	$-2.2398648 \cdot 10^{9}$	$1.5047064 \cdot 10^{7}$	$-1.9443066 \cdot 10^{6}$	$2.2657973 \cdot 10^{4}$
0.993422	0.9938748	$9.6792057 \cdot 10^{9}$	$1.013336 \cdot 10^{7}$	$-1.9258934 \cdot 10^{6}$	$2.1243369 \cdot 10^{4}$
0.9938748	0.9941958	$-1.1321291 \cdot 10^{10}$	$2.3280141 \cdot 10^{7}$	$-1.9107655 \cdot 10^{6}$	$2.0374397 \cdot 10^{4}$
0.9941958	0.994321	$-1.1167542 \cdot 10^{10}$	$1.2377738 \cdot 10^{7}$	$-1.8993193 \cdot 10^{6}$	$1.9763065 \cdot 10^{4}$
0.994321	0.9951908	$1.2143136 \cdot 10^{10}$	$8.1815344 \cdot 10^{6}$	$-1.8967442 \cdot 10^{6}$	$1.9525348 \cdot 10^{4}$
0.9951908	0.9967765	$-5.061124 \cdot 10^{9}$	$3.9866011 \cdot 10^{7}$	$-1.8549549 \cdot 10^{6}$	$1.7889833 \cdot 10^{4}$
0.9967765	0.997357	$-2.2037852 \cdot 10^{10}$	$1.5788979 \cdot 10^{7}$	$-1.7667 \cdot 10^{6}$	$1.5028404 \cdot 10^{4}$
0.997357	0.9973813	$1.2526581 \cdot 10^{11}$	$-2.2589941 \cdot 10^{7}$	$-1.7706479 \cdot 10^{6}$	$1.4003844 \cdot 10^{4}$
0.9973813	0.9974038	$1.1684825 \cdot 10^{11}$	$-1.3476853 \cdot 10^{7}$	$-1.7715226 \cdot 10^{6}$	$1.3960894 \cdot 10^{4}$
0.9974038	0.9976908	$-4.1265222 \cdot 10^{10}$	$-5.5895969 \cdot 10^{6}$	$-1.7719516 \cdot 10^{6}$	$1.392103 \cdot 10^{4}$
0.9976908	0.9980265	$-3.6644545 \cdot 10^{10}$	$-4.1118953 \cdot 10^{7}$	$-1.7853569 \cdot 10^{6}$	$1.3411044 \cdot 10^{4}$
0.9980265	0.998114	$8.3659854 \cdot 10^{10}$	$-7.802917 \cdot 10^{7}$	$-1.8253609 \cdot 10^{6}$	$1.2805588 \cdot 10^{4}$
0.998114	0.9983623	$-1.3385974 \cdot 10^{11}$	$-5.6068459 \cdot 10^{7}$	$-1.8370944 \cdot 10^{6}$	$1.2645327 \cdot 10^{4}$
0.9983623	0.998607	$-3.5568462 \cdot 10^{10}$	$-1.557605 \cdot 10^{8}$	$-1.889681 \cdot 10^{6}$	$1.2183765 \cdot 10^{4}$
0.998607	0.9986313	$3.0784292 \cdot 10^{11}$	$-1.8187665 \cdot 10^{8}$	$-1.9723177 \cdot 10^{6}$	$1.1711414 \cdot 10^{4}$
0.9986313	0.9986538	$-2.4671884 \cdot 10^{11}$	$-1.5948107 \cdot 10^{8}$	$-1.9805956 \cdot 10^{6}$	$1.1663483 \cdot 10^{4}$
0.9986538	0.9992765	$-2.2464418 \cdot 10^{11}$	$-1.761346 \cdot 10^{8}$	$-1.9881469 \cdot 10^{6}$	$1.1618836 \cdot 10^{4}$
0.9992765	0.9992985	$-3.3438978 \cdot 10^{12}$	$-5.9582608 \cdot 10^{8}$	$-2.4688855 \cdot 10^{6}$	$1.0258155 \cdot 10^{4}$
0.9992985	0.999364	$-1.5352094 \cdot 10^{12}$	$-8.1652333 \cdot 10^{8}$	$-2.4999571 \cdot 10^{6}$	$1.0203515 \cdot 10^{4}$
0.999364	0.9994285	$-5.0645096 \cdot 10^{11}$	$-1.118192 \cdot 10^{9}$	$-2.626681 \cdot 10^{6}$	$1.0035834 \cdot 10^{4}$

0.9994285	0.9994865	$1.5025164 \cdot 10^{12}$	$-1.2161902 \cdot 10^{9}$	$-2.7772486 \cdot 10^{6}$	$9.8616247 \cdot 10^{3}$
0.9994865	0.9995485	$1.2879654 \cdot 10^{12}$	$-9.5475238 \cdot 10^{8}$	$-2.9031633 \cdot 10^{6}$	$9.6967462 \cdot 10^{3}$
0.9995485	0.9995773	$-9.2000902 \cdot 10^{11}$	$-7.1519081 \cdot 10^{8}$	$-3.0066998 \cdot 10^{6}$	$9.513387 \cdot 10^{3}$
0.9995773	0.9996123	$-5.4278771 \cdot 10^{12}$	$-7.9454159 \cdot 10^{8}$	$-3.0501046 \cdot 10^{6}$	$9.4263313 \cdot 10^{3}$
0.9996123	0.9996785	$-4.0122689 \cdot 10^{12}$	$-1.3644687 \cdot 10^{9}$	$-3.12567 \cdot 10^{6}$	$9.3183716 \cdot 10^{3}$
0.9996785	0.9997023	$7.379071 \cdot 10^{12}$	$-2.1619071 \cdot 10^{9}$	$-3.3592924 \cdot 10^{6}$	$9.1041406 \cdot 10^{3}$
0.9997023	0.9997365	$3.5562138 \cdot 10^{13}$	$-1.6361483 \cdot 10^{9}$	$-3.4494962 \cdot 10^{6}$	$9.0232368 \cdot 10^{3}$
0.9997365	0.9997985	$5.37386 \cdot 10^{13}$	$2.0178613 \cdot 10^{9}$	$-3.4364225 \cdot 10^{6}$	$8.9046011 \cdot 10^{3}$
0.9997985	0.9998273	$-1.5309891 \cdot 10^{14}$	$1.2013241 \cdot 10^{10}$	$-2.5664942 \cdot 10^{6}$	$8.7121069 \cdot 10^{3}$
0.9998273	0.999857	$-4.2740774 \cdot 10^{14}$	$-1.1915398 \cdot 10^{9}$	$-2.2553703 \cdot 10^{6}$	$8.6446117 \cdot 10^{3}$
0.999857	0.9998813	$3.0511372 \cdot 10^{13}$	$-3.9337681 \cdot 10^{10}$	$-3.4611146 \cdot 10^{6}$	$8.565206 \cdot 10^{3}$
0.9998813	0.9999038	$5.3460507 \cdot 10^{14}$	$-3.7117978 \cdot 10^{10}$	$-5.3151643 \cdot 10^{6}$	$8.458576 \cdot 10^{3}$
0.9999038	0.9999285	$2.0758213 \cdot 10^{14}$	$-1.0321365 \cdot 10^{9}$	$-6.1735419 \cdot 10^{6}$	$8.3262834 \cdot 10^{3}$
0.9999285	0.9999523	$4.2080678 \cdot 10^{13}$	$1.4380836 \cdot 10^{10}$	$-5.8431616 \cdot 10^{6}$	$8.1760031 \cdot 10^{3}$
0.9999523	0.9999865	$-3.9223533 \cdot 10^{13}$	$1.7379085 \cdot 10^{10}$	$-5.0888634 \cdot 10^{6}$	$8.0459034 \cdot 10^{3}$
0.9999865	1.0000485	$-4.0699553 \cdot 10^{13}$	$1.3348867 \cdot 10^{10}$	$-4.0364311 \cdot 10^{6}$	$7.8904207 \cdot 10^{3}$
1.0000485	1.0000773	$-3.6445618 \cdot 10^{13}$	$5.7787499 \cdot 10^{9}$	$-2.8505189 \cdot 10^{6}$	$7.6817752 \cdot 10^{3}$
1.0000773	1.0001675	$-1.1291565 \cdot 10^{13}$	$2.6353154 \cdot 10^{9}$	$-2.6086145 \cdot 10^{6}$	$7.6037332 \cdot 10^{3}$
1.0001675	1.0001785	$2.3744333 \cdot 10^{13}$	$-4.2187597 \cdot 10^{8}$	$-2.4088516 \cdot 10^{6}$	$7.3814702 \cdot 10^{3}$
1.0001785	1.0002365	$1.0221445 \cdot 10^{13}$	$3.6168703 \cdot 10^{8}$	$-2.4095137 \cdot 10^{6}$	$7.3549534 \cdot 10^{3}$
1.0002365	1.0002703	$1.3767596 \cdot 10^{13}$	$2.1402185 \cdot 10^{9}$	$-2.2644031 \cdot 10^{6}$	$7.2184126 \cdot 10^{3}$
1.0002703	1.0002925	$9.8935884 \cdot 10^{12}$	$3.5341876 \cdot 10^{9}$	$-2.0728919 \cdot 10^{6}$	$7.1449561 \cdot 10^{3}$
1.0002925	1.0002985	$4.4489743 \cdot 10^{13}$	$4.1945847 \cdot 10^{9}$	$-1.9009267 \cdot 10^{6}$	$7.1006929 \cdot 10^{3}$
1.0002985	1.0004175	$-4.1658653 \cdot 10^{11}$	$4.9954 \cdot 10^{9}$	$-1.8457868 \cdot 10^{6}$	$7.089448 \cdot 10^{3}$
1.0004175	1.0004285	$-2.9717144 \cdot 10^{13}$	$4.8466786 \cdot 10^{9}$	$-6.7457948 \cdot 10^{5}$	$6.9398372 \cdot 10^{3}$
1.0004285	1.0004865	$-1.4444854 \cdot 10^{13}$	$3.8660129 \cdot 10^{9}$	$-5.7873987 \cdot 10^{5}$	$6.9329637 \cdot 10^{3}$
1.0004865	1.0005203	$-1.3288582 \cdot 10^{13}$	$1.3526083 \cdot 10^{9}$	$-2.7605984 \cdot 10^{5}$	$6.9095837 \cdot 10^{3}$
1.0005203	1.0005265	$-1.8551767 \cdot 10^{13}$	$7.1393744 \cdot 10^{6}$	$-2.3016836 \cdot 10^{5}$	$6.9012965 \cdot 10^{3}$
1.0005265	1.0005425	$1.1857911 \cdot 10^{13}$	$-3.4070626 \cdot 10^{8}$	$-2.3225315 \cdot 10^{5}$	$6.8998537 \cdot 10^{3}$
1.0005425	1.000614	$-5.8000946 \cdot 10^{12}$	$2.2847346 \cdot 10^{8}$	$-2.3404888 \cdot 10^{5}$	$6.896099 \cdot 10^{3}$
1.000614	1.0006675	$2.0143887 \cdot 10^{13}$	$-1.0156468 \cdot 10^{9}$	$-2.9033177 \cdot 10^{5}$	$6.8784124 \cdot 10^{3}$
1.0006675	1.0007703	$-7.2308314 \cdot 10^{12}$	$2.2174471 \cdot 10^{9}$	$-2.2603546 \cdot 10^{5}$	$6.8630573 \cdot 10^{3}$
1.0007703	1.0008623	$5.0089928 \cdot 10^{10}$	$-1.1456696 \cdot 10^{7}$	$6.3005486 \cdot 10^{2}$	$6.8553991 \cdot 10^{3}$
1.0008623	1.0010203	$-6.7324916 \cdot 10^{9}$	$2.3681243 \cdot 10^{6}$	$-2.0609371 \cdot 10^{2}$	$6.8553991 \cdot 10^{3}$
1.0010203	1.001107	$4.4352981 \cdot 10^{9}$	$-8.2307674 \cdot 10^{5}$	$3.8023795 \cdot 10^{1}$	$6.8553991 \cdot 10^{3}$
1.001107	1.0011313	$-5.7581594 \cdot 10^{9}$	$3.3120959 \cdot 10^{5}$	$-4.6456802 \cdot 10^{0}$	$6.8553991 \cdot 10^{3}$
1.0011313	1.0011538	$1.4096984 \cdot 10^{9}$	$-8.7696511 \cdot 10^{4}$	$1.2595118 \cdot 10^{0}$	$6.8553991 \cdot 10^{3}$
1.0011538	1.0012703	$-2.3800009 \cdot 10^{7}$	$7.4581314 \cdot 10^{3}$	$-5.458517 \cdot 10^{-1}$	$6.8553991 \cdot 10^{3}$
1.0012703	1.0017765	$8.2924545 \cdot 10^{5}$	$-8.5997167 \cdot 10^{2}$	$2.2283391 \cdot 10^{-1}$	$6.8553991 \cdot 10^{3}$
1.0017765	1.001821	$-3.7696179 \cdot 10^{6}$	$3.9944486 \cdot 10^{2}$	$-1.0307792 \cdot 10^{-2}$	$6.8553991 \cdot 10^{3}$
1.001821	1.001864	$8.7232639 \cdot 10^{5}$	$-1.0379913 \cdot 10^{2}$	$2.8484428 \cdot 10^{-3}$	$6.8553991 \cdot 10^{3}$
1.001864	1.0021123	$-1.5056455 \cdot 10^{4}$	$8.7309761 \cdot 10^{0}$	$-1.2394877 \cdot 10^{-3}$	$6.8553991 \cdot 10^{3}$
1.0021123	1.002357	$4.9369419 \cdot 10^{3}$	$-2.4823188 \cdot 10^{0}$	$3.1174145 \cdot 10^{-4}$	$6.8553991 \cdot 10^{3}$
1.002357	1.0023813	$-2.004318 \cdot 10^{4}$	$1.1426308 \cdot 10^{0}$	$-1.6147184 \cdot 10^{-5}$	$6.8553991 \cdot 10^{3}$
1.0023813	1.0024038	$4.7812066 \cdot 10^{3}$	$-3.1551051 \cdot 10^{-1}$	$3.9104829 \cdot 10^{-6}$	$6.8553991 \cdot 10^{3}$
1.0024038	1.003114	$-4.2671381 \cdot 10^{0}$	$7.2209382 \cdot 10^{-3}$	$-3.0260325 \cdot 10^{-6}$	$6.8553991 \cdot 10^{3}$
1.003114	1.0033623	$2.2975848 \cdot 10^{0}$	$-1.8712663 \cdot 10^{-3}$	$7.7357194 \cdot 10^{-7}$	$6.8553991 \cdot 10^{3}$
1.0033623	1.005922	$2.2115351 \cdot 10^{-2}$	$-1.6014005 \cdot 10^{-4}$	$2.6927531 \cdot 10^{-7}$	$6.8553991 \cdot 10^{3}$
1.005922	1.0066958	$1.0157171 \cdot 10^{-2}$	$9.6892578 \cdot 10^{-6}$	$-1.1584111 \cdot 10^{-7}$	$6.8553991 \cdot 10^{3}$
1.0066958	1.0126248	$-2.7111595 \cdot 10^{-3}$	$3.3266592 \cdot 10^{-5}$	$-8.2604017 \cdot 10^{-8}$	$6.8553991 \cdot 10^{3}$
1.0126248	1.0191958	$1.3192554 \cdot 10^{-3}$	$-1.4956802 \cdot 10^{-5}$	$2.5954727 \cdot 10^{-8}$	$6.8553991 \cdot 10^{3}$
1.0191958	1.0313748	$-8.7896014 \cdot 10^{-4}$	$1.1049679 \cdot 10^{-5}$	$2.8102735 \cdot 10^{-10}$	$6.8553991 \cdot 10^{3}$
1.0313748	1.0498763	$1.4794156 \cdot 10^{-3}$	$-2.1064887 \cdot 10^{-5}$	$-1.2169419 \cdot 10^{-7}$	$6.8553991 \cdot 10^{3}$
1.0498763	1.0500013	$1.4794156 \cdot 10^{-3}$	$6.1049335 \cdot 10^{-5}$	$6.1807807 \cdot 10^{-7}$	$6.8553991 \cdot 10^{3}$

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